A novel permit scheme for managing parking competition and bottleneck congestion

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ABSTRACT

Morning commuters may have to depart from home earlier to secure a parking space when parking supply in the city center is insufficient. Recent studies show that parking reservations can reduce highway congestion and deadweight loss of parking competition simultaneously. This study develops a novel tradable parking permit scheme to realize or implement parking reservations when commuters are either homogeneous or heterogeneous in their values of time. It is found that an expirable parking permit scheme with an infinite number of steps, i.e., the ideal-scheme, is superior to a time-varying pricing scheme in the sense that designing a permit scheme does not require commuters’ value of time information and the performance of the scheme is robust to the variation of commuters’ value of time. Although it is impractical to implement the ideal-scheme with an infinite number of steps, the efficiency loss of a permit scheme with finite steps can be bounded in both cases of homogeneous and heterogeneous commuters. Moreover, considering the permit scheme may lead to an undesirable benefit distribution among commuters, we propose an equal cost-reduction distribution of parking permits where auto commuters with higher value of time will receive fewer permits.

1. Introduction

Parking policy has been known for long as a powerful and efficient tool for transportation management (Young et al., 1991). In a static stochastic traffic assignment model, Bifulco (1993) introduced parking search times, types and fees for the evaluation of various parking policies. Arnott and co-authors have developed a series of economic analyses of downtown parking over the years (e.g., Arnott et al., 1991; Arnott and Rowse, 2009; Arnott and Inci, 2010). Besides, many researchers have investigated parking fee as an instrument to help manage traffic. (e.g., Glazer and Niskanen, 1992; Verhoef et al., 1995; Anderson and de Palma, 2004; Zhang et al., 2008; Qian et al., 2012; Fosgerau and de Palma, 2013).

Qian et al. (2011) pointed out that parking availability will affect commuters’ choices of travel mode, route and departure time. In the same spirit of tradable mobility credits proposed by Yang and Wang (2011), Zhang et al. (2011) proposed a parking permits distribution and trading scheme for managing the morning commute when parking supply in the downtown is insufficient. More recently, Yang et al. (2013) proved that, when the parking supply is inadequate to accommodate the potential auto demand, retaining some parking spaces open for competition while keeping others for reservation can smooth out commuters’ arrivals to the highway bottleneck and thus reduce the total system cost. For the same problem, an
accompanying paper, Liu et al. (2014), proposed a more elaborate parking reservation scheme where reservations may expire at pre-determined times. They showed that if differentiated expiration times are properly designed, the morning peak will be further smoothed out and the total social cost can be further reduced.

In this paper, we propose a novel tradable parking permit scheme to realize parking reservations of commuters, and thus to manage both parking competition and bottleneck congestion. The proposed permit scheme is different from that in Zhang et al. (2011) on several aspects: first, parking reservations considered here can be either expirable or non-expirable, thus the corresponding parking permits for reservation would also be either expirable or non-expirable. Secondly, this study considers the socially optimal supply of parking permits (as well as parking reservation) while Zhang et al. (2011) only considers reserving all available parking spaces to commuters. Also, we extend the studies by Yang et al. (2013) and Liu et al. (2014) by taking into account commuters’ heterogeneity in their value of time (VOT). In reality, people’s VOT varies, and commuters with different VOT may respond to the parking space constraint and the parking permit scheme differently. Lastly, since commuters are heterogeneous, a parking permit scheme may lead to an undesirable distribution of benefit among the commuters. Such equity issues has been observed in previous studies when considering congestion pricing with both static models (e.g., Glazer, 1981; Evans, 1992) and dynamic models (e.g., Cohen, 1987; Arnott et al., 1994; Xiao and Zhang, 2013). Efforts have been made to achieve better equity (e.g., Wu et al., 2012; Xiao et al., 2013). In this study, we would also explore the distribution of parking permits to achieve a more equitable benefit distribution among commuters.

We consider that two competing modes, a highway and a parallel transit line, connect a residential area to the Central Business District (CBD). For the highway, bottleneck congestion is considered in order to make our model more tractable. Vickrey (1969) firstly introduced the bottleneck model of congestion dynamics. Smith (1984) and Daganzo (1985) then established the existence and uniqueness of the time-dependent equilibrium distribution of arrivals at a single bottleneck respectively. Arnott et al. (1990) provided a thorough economic analysis of the bottleneck model and showed that congestion tolls can generate efficiency gains by altering the frequency distribution of departure times. The bottleneck model has been adopted to study various issues. For recent comprehensive reviews, see, e.g., Arnott et al. (1998) and de Palma and Fosgerau (2011).

The remainder of this paper is organized as follows. Section 2 considers the bi-modal equilibrium with homogeneous commuters, and examines tradable non-expirable and expirable parking permit schemes. In Sections 3, commuters’ heterogeneity in VOT is introduced, and then the effects of tradable non-expirable and expirable parking permit schemes are discussed. In addition, welfare effects are explored and the equal cost-reduction permit distribution is introduced. Section 4 discusses road pricing to eliminate bottleneck congestion and deadweight loss of parking competition, and then compares it with the tradable expirable parking permit scheme. Finally, Section 5 concludes the paper and provides some discussions.

2. The case with homogeneous commuters

2.1. Basic setting

Suppose a fixed number of $N$ commuters travel from home to the CBD every morning, and there are two competing modes: a bottleneck-constrained highway and a parallel transit line. We assume that all parking spaces are located at the CBD, and the walking time from parking spaces to workplace is ignored without loss of generality. The numbers of auto and transit commuters are denoted by $N_a$ and $N_b$ respectively, and $N = N_a + N_b$.

The generalized travel cost of an auto commuter consists of his or her fixed money cost, travel time cost and schedule delay cost. Departing at time $t$, the travel cost is given by

$$c_a(t) = p_a + x \cdot T_a(t) + \beta(t' - t - T_a(t)),$$

where $p_a$ is the fixed money cost; $T_a(t)$ is the travel time; $t'$ is the desired arrival time at the destination, and $x$ and $\beta$ are the value of travel time and the marginal cost of early arrival respectively. Without loss of generality, the free-flow commute time is assumed to be zero, and thus $T_a(t)$ only contains the queuing time, i.e., $T_a(t) = q(t)/s$, where $q(t)$ is the queue length at the highway bottleneck at time $t$, and $s$ is the service capacity of the bottleneck. In addition, according to empirical evidences and for the existence and uniqueness of the equilibrium, it is assumed that $x > \beta > 0$. Also note that in this study, we assume late arrivals of commuters are not allowed and a discussion on the relaxation of this assumption will be presented in the last section.

For transit commuters, the travel cost consists of travel time cost and transit fare. The generalized travel cost of commuters by transit can be expressed as

$$c_b = p_b + x T_b,$$

where $p_b$ is the transit fare and $T_b$ is the constant travel time on transit. Note that since we assume zero free-flow time on highway, $T_b$ indeed represents additional travel time on transit compared with highway. It is assumed that $p_a > p_b$, which means the fixed money cost of the auto mode is higher than the transit fare. Note that the schedule delay cost of transit commuters is ignored for simplicity since we want to focus on commuters’ responses to the insufficient parking supply and auto mode.
2.1.1. Bi-modal equilibrium without parking space constraint

For the homogeneous case, we assume \( p_a < c_b = p_b + xT_b \) and \( p_a + \beta N/s > c_b = p_b + xT_b \). This would lead to an interior equilibrium at which both auto and transit are used for commuting. The numbers of auto and transit commuters can be determined as \( N_a = s(p_b + xT_b - p_a)/\beta \) and \( N_b = N - N_a \). Note that at equilibrium the auto travel cost, \( c_a = p_a + \beta N_a/s \), is identical to the transit travel cost, \( c_b = p_b + xT_b \). Fig. 1(a) shows the auto commuting equilibrium, where \( s \) is the bottleneck capacity, \( r = \frac{x}{\beta} s \) is the departure rate from home for commuters.

2.1.2. Bi-modal equilibrium with parking space constraint

When the total parking supply at CBD, denoted as \( M \), is unable to accommodate all the potential auto demand, i.e., \( N_a \), auto commuters have to depart earlier to compete for parking spaces on a first-come-first-served basis. At the bi-modal equilibrium, the travel cost of auto commuters will be identical to that of transit commuters, which is given by Eq. (2). Fig. 1(b) depicts the auto commuting equilibrium. It can be seen that the last auto commuter arrives at the destination before \( t^* \) due to the competition for limited parking spaces. \( (t_u < t^*) \).

2.2. Equilibrium under non-expirable parking permit scheme

Suppose the city government manages all downtown parking spaces via a tradable parking permit scheme. Specifically, every day a certain number of parking permits are distributed to commuters, i.e., \( M' \) where \( M' \in [0, M] \). A commuter with a parking permit can use it to reserve a parking space in the CBD for his or her morning commute. We assume parking permits are allowed to be traded among commuters in a perfect competition market without any transaction cost. For ease of presentation, we denote auto commuters using a parking permit to reserve a parking space to be \( r \)-commuters, and denote the other as \( u \)-commuters. Note that the number of \( u \)-commuters is equal to \( M - N_a \).

Fig. 2(a) depicts the auto commuting equilibrium under the given permit scheme with \( M' \) permits distributed per day. The equilibrium permit price can be expressed as follows:

\[
p_{ne} = p_b + xT_b - \left( p_a + \beta \frac{M'}{s} \right).
\]  

(3)

The permit price reflects the benefit from parking reservation as compared to competing for a parking space or taking transit. Note that the subscript \( ne \) corresponds to the non-expirable permit scheme. Suppose that for a given period, the average number of permits distributed to the \( m \)-th commuter is \( n_m \) per day, his or her average travel cost considering the permits can be expressed as:

\[
c_{ne}(m) = p_b + xT_b - n_mp_{ne}.
\]  

(4)

By minimizing the total travel cost (the permit price is excluded), we can determine the unique socially optimal number of permits (i.e., the number of reservations) should be distributed per day as follows:

\[
M' = \begin{cases} 
\frac{1}{2}N_a & \frac{1}{2}N_a \leq M < \frac{3}{2}N_a \\
M & M < \frac{1}{2}N_a
\end{cases}.
\]  

(5)

Eq. (5) indicates that, when the parking supply is severely insufficient (less than half of the demand), all the parking spaces should be reserved to commuters via parking permits; when the parking supply is insufficient, but more than half of the potential auto demand, the number of reserved spaces should be equal to half of the potential demand. This result is consistent with that in Yang et al. (2013). From the fact that \( N_a = s(p_b + xT_b - p_a)/\beta \) and Eq. (3), we have the following observation.
Observation 1. For the homogeneous case, the optimal design of the non-expirable scheme requires commuters’ VOT information, besides the total parking supply and highway bottleneck capacity.

2.3. Equilibrium under expirable parking permit scheme

Recently, Liu et al. (2014) considers expirable parking reservations. In their case, since parking reservation will expire after a certain time, the commuters’ departure time choices will not only be governed by the bottleneck capacity and their schedule preference, but also the expiration time of parking reservation. Therefore, commuters’ departure might be dispersed due to differentiated expiration times.

In this study, we now introduce the expirable parking permit to realize commuters’ expirable parking reservations to form the socially optimal departure pattern. Without loss of generality, suppose parking spaces are divided into \( k \) groups equally, i.e., \( M_i^r = \frac{M}{k} \), where \( i = 1, 2, \ldots, k \). Reservations for parking spaces in the \( i \)-th group will expire at \( t_i \), where \( i = 1, 2, \ldots, n \) and \( t_1 < t_{i+1} \). Correspondingly, there are \( k \) classes of permits and the \( i \)-th class of permits will be used to make reservations that expire at \( t_i \). For given \( M' \) and \( k \), under the optimal design of the permit scheme, arrivals of \( r \)-commutes should be consecutive such that there is no capacity waste, which implies \( t_i - t_{i-1} = \frac{M}{kT_b} \) for \( i \geq 2 \), and the schedule delay cost is also minimized, i.e., \( t_k = t' \).

Fig. 2(b) depicts the auto commuting equilibrium under such an expirable permit scheme with \( M' = M \) and given \( k \).

When \( k \) approaches infinity, the queue at the bottleneck and the competition for parking can be completely eliminated, and the equilibrium price of a parking permit with expiration time \( t \) can be determined as follows:

\[
\hat{p}_e(t) = p_b - p_a + \alpha T_b - \beta (t' - t), \tag{6}
\]

where \( t \in [t' - \frac{M}{k}, t'] \). Note that the subscript \( e \) corresponds to an expirable permit scheme. Differentiating Eq. (6) with respect to \( t \), we have:

\[
\hat{p}_e(t) = \beta > 0. \tag{7}
\]

Thus we know that the price of the expirable parking permit is an strictly increasing function of its expiration time, \( t \). This is consistent with our intuition that the permit with a later expiration time is more valuable. For ease of presentation, we refer to the optimal expirable permit scheme with \( M' = M \) and \( k \to \infty \) as the ideal-scheme since it completely eliminates the queue at the bottleneck and competition for limited parking spaces.

Observation 2. For the homogeneous case, the optimal design of the ideal-scheme with \( M' = M \) and \( k \to \infty \) does not require commuters’ VOT information. Further, this ideal-scheme is robust to commuters’ VOT variation.

Knowing the total parking supply and the highway capacity, the system planner can design the ideal-scheme, in the absence of the information on commuters’ VOT. Further, the scheme is robust to commuters’ VOT variation, i.e., the equilibrium flow pattern under the scheme will not change for commuters with different levels of VOT \( \{x \} \) and \( \beta \). This makes the scheme quite appealing since in reality it is certainly difficult to obtain VOT of commuters accurately. On the other hand, it is also impractical to implement such a scheme with \( k \to \infty \). However, as shown below, a scheme with a reasonably large \( k \) yields a small loss of efficiency in total social cost reduction.

Since for transit commuters the total cost is constant, we focus on the travel cost associated with the auto mode, and hereinafter, the total social cost, if not mentioned separately, is for auto mode only. Under the scheme with \( M' = M \) and \( k \) steps, the total social cost can be expressed as follows:
where $k \geq 1$ (to save space, we omit the detailed derivation). The subscript $ho$ represents the case with homogeneous commuters. We then define the measure of percentage efficiency loss as follows:

$$l_{ho}(k) = \frac{TC_{ho}(k) - TC_{ho}(\infty)}{TC_{ho} - TC_{ho}(\infty)},$$

where $TC_{ho}(\infty)$ is the total cost under the ideal-scheme where $k \to \infty$, and $TC_{ho}$ is the total cost when no scheme is introduced. Then, we see that in Eq. (9), $TC_{ho} - TC_{ho}(\infty)$ is the maximum cost can be reduced by implementing the ideal-scheme, and $TC_{ho}(k) - TC_{ho}(\infty)$ is the efficiency loss caused by the scheme with $k$ steps.

**Proposition 1.** For the homogeneous case, the efficiency loss $l_{ho}(k) < \frac{1}{e}$.

**Proof.** Firstly, note that the parking space constraint implies that

$$M < \frac{S}{\beta} (p_a + \alpha T_b - p_a) \Rightarrow p_b + \alpha T_b > p_a + \beta \frac{M}{S}.$$  \hspace{1cm} (10)

From Eqs. (8)–(10), it can be verified that

$$l_{ho}(k) = \left(\frac{p_a + \frac{k+1}{2k} \beta \frac{M}{S}}{p_a + \alpha T_b} - \frac{p_a + \frac{k+1}{2k} \beta \frac{M}{S}}{p_a + \alpha T_b} \right) < \frac{\beta M}{s} - \frac{\beta M}{s} \frac{(k-1)}{k}.$$  \hspace{1cm} (11)

This completes the proof. □

According to Proposition 1, a permit scheme with 10 steps would only lose efficiency less than 10% when compared with the ideal-scheme. Note that in the above efficiency loss analysis of the scheme with $k$ steps, we set $M' = M$, which may not be optimal when $k$ is not large enough, as shown in Liu et al. (2014).

In reality, it is possible that the parking management authority may want to retain some parking spaces unreserved to remain certain level of management flexibility, i.e., $M'$ is set to be less than $M$. Suppose, under a scheme with $k$ steps, we set $M' = \lambda M$ where $\lambda \in (0, 1]$. It is possible that a $\lambda < 1$ may lead to lower total social cost than that under $\lambda = 1$. However, if $k$ is large enough, setting $\lambda < 1$ would lead to an efficiency loss. In the following analysis, we will show this efficiency loss can be bounded.

We begin with expressing the total social cost under the optimal scheme with $k$ steps and given $M'$ as follows:

$$TC_{ho}(k, M') = (p_b + \alpha T_b)(M - M') + \left(p_a + \left(\frac{k+1}{2k} \frac{M}{s}\right)M'\right).$$  \hspace{1cm} (12)

where $k \geq 1$. The subscript $ho$ represents the case with homogeneous commuters. Then we notice that Eq. (8) is a special instance of Eq. (12) where $M' = M$. Note that when $TC_{ho}(k, \lambda M) \leq TC_{ho}(k, M)$, the scheme with $M' = \lambda M$ would lead to efficiency higher than or equal to that with $M' = M$, implying there is no efficiency loss; otherwise, we define the percentage efficiency loss measure:

$$l_{ho}(\lambda) = \frac{TC_{ho}(k, \lambda M) - TC_{ho}(k, M)}{TC_{ho} - TC_{ho}(k, M)}.$$  \hspace{1cm} (13)

In Eq. (13), $TC_{ho} - TC_{ho}(k, M)$ is the travel cost can be reduced under the optimal scheme with $k$ steps and $M' = M$, and $TC_{ho}(k, \lambda M) - TC_{ho}(k, M)$ is the portion that the optimal scheme with $k$ steps and $M' = \lambda M$ is unable to reduce compared to the scheme with $M' = M$.

**Proposition 2.** For the homogeneous case, the efficiency loss $l_{ho}(\lambda) < 1 - \lambda$.

**Proof.** Firstly, note that the parking space constraint implies that

$$M < \frac{S}{\beta} (p_a + \alpha T_b - p_a) \Rightarrow p_b + \alpha T_b > p_a + \beta \frac{M}{S}.$$  \hspace{1cm} (14)

From Eqs. (8) and (13), it can be verified that

$$l_{ho}(\lambda) = \lambda (1 - \lambda) \left(\frac{p_a + \alpha T_b + \lambda (p_a + \frac{k+1}{2k} \beta \frac{M}{s})}{p_a + \alpha T_b} - \frac{p_a + \frac{k+1}{2k} \beta \frac{M}{s}}{p_a + \frac{k+1}{2k} \beta \frac{M}{s}}\right).$$  \hspace{1cm} (15)
Rearranging Eq. (15), we have
\[
\kappa_{k}(\lambda) = (1 - \lambda) - \frac{\lambda(1 - \lambda)(kB + M)}{(p_b + xT_b) - (p_b + kB + \beta M)}, \tag{16}
\]

From Eqs. (14) and (16), and the fact that \(k \geq 1\) we know that \(\kappa_{k}(\lambda) < (1 - \lambda)\) for \(\lambda < 1\). \(\square\)

According to Proposition 2, a permit scheme with \(k\) steps and \(M^* = 0.9M\) would only yield an efficiency loss of less than 10% when compared with the scheme with \(k\) steps and \(M^* = M\). Denote the efficiency loss of the optimal scheme with \(k\) steps and \(M^* = M\) compared to the ideal-scheme by \(e_{k}(t, \lambda)\), then further from Proposition 1, we see that \(e_{k}(t, \lambda)\) is less than 1 - \(\lambda(1 - \frac{1}{k})\). This means, for a scheme with 10 steps and 90% percent of parking spaces reserved, the combined efficiency loss would be less than 1 - 90% \times (1 - 1/10) = 19%.

2.4. Permit distribution and equity

Suppose the number of permits with expiration time \(\bar{t}\) distributed to the \(m\)-th commuter is \(n_m(\bar{t})\) per day, his or her average travel cost considering the permits can be expressed as:
\[
c_e(m) = p_b + xT_b - \int_{\bar{t}}^{T} n_m(\bar{t})p_e(\bar{t})d\bar{t}. \tag{17}
\]

Due to the tradability of the parking permit (without any transaction cost), the initial permit distribution will not influence the commuting equilibrium. This implies that for a given day, once the number of parking reservations is determined, the equilibrium flow pattern at the bottleneck will be identical. However, the initial permit distribution will affect the welfare allocation among commuters, which might cause unfairness among commuters.

The most natural way of distributing permits is perhaps to distribute them equally among commuters (uniform permit distribution). At this stage, since only homogeneous commuters are considered, inequity issue does not exist. For the non-expirable parking permit scheme, once we distribute the permits to everyone equally, i.e., \(m_i = \frac{m_N}{N}\), the individual travel cost given by Eq. (4) will be identical, thus the monetary cost reduction is the same, i.e., an equitable reduction. This result can be easily extended to the expirable permit scheme, i.e., by distributing equally each class of permits to commuters, the permit distribution will result in equal reduction of travel cost. However, we see that as long as the third term in Eq. (17), i.e., the benefit from trading permits, is identical, the permit distribution will be equitable in the sense of monetary cost reduction, and we do not need to distribute each class of permits equally. This allows additional flexibility for permit distribution, and a commuter can choose to receive either fewer permits with a higher price or more permits but with a lower price.

3. The case with heterogeneous commuters

3.1. Basic setting

Now we turn to consider the case with heterogeneous commuters, as in reality VOT varies among commuters. Generally, high-income people are more willing to buy more permits with higher price so as to save their travel cost and schedule delay cost. Similar to previous studies considering commuters’ heterogeneity in VOT (e.g., Vickrey, 1973; Xiao et al., 2011, 2013; Xiao and Zhang, 2013), we assume the ratio between the unit value of schedule delay and travel time is identical for all commuters, i.e., \(\beta = \rho \alpha\), where \(\rho\) is constant and 0 < \(\rho\) < 1. Note that some other studies considered different types of user heterogeneity (e.g., van den Berg and Verhoef, 2011; Tian et al., 2013). Further, commuters’ VOT, i.e., \(\alpha\), continuously increases from \(\alpha\) to \(\bar{\alpha}\) and follows the following cumulative distribution function:
\[
F(x) = \Pr\{\alpha \leq x\}. \tag{18}
\]

For convenience of further analysis, we arrange commuters in a decreasing order of \(\alpha\) and thus \(\alpha(m)\) specifies the \(m\)-th commuter’s VOT, which can be given as follows:
\[
\alpha(m) = F^{-1}\left(1 - \frac{m}{N}\right), \tag{19}
\]

where \(N\) is the total number of commuters traveling to the CBD, and \(m \leq N\). The numbers of auto and transit commuters are still denoted by \(N_a\) and \(N_t\) respectively, and \(N = N_a + N_t\).

Under the above setting of user heterogeneity, without parking permits, similar to Eq. (1), the generalized travel cost of the \(m\)-th commuter by taking auto mode and departing at time \(t\) is
\[
c_a(m, t) = p_a + \alpha(m)(T_a(t) + \rho(t - T_a(t))). \tag{20}
\]
where \(p_a\) is the fixed money cost, \(T_a(t)\) is the travel time and \(\alpha(m)\) is the VOT for the \(m\)-th commuter. For transit commuters, similar to Eq. (2), the generalized travel cost of the \(m\)-th commuter is
\[
c_b(m) = p_b + \alpha(m)T_b, \tag{21}
\]
where \(p_b\) is the transit fare and \(T_b\) is the constant travel time on transit.
3.1.1. Bi-modal equilibrium without parking space constraint

For the heterogeneous case, to ensure an interior equilibrium, we assume that \( p_a < c_b = p_b + T_Tb \) and \( p_a + \frac{z \rho \lambda}{s} > c_b = p_b + T_Tb \). The numbers of auto commuters, \( \tilde{N}_a \), then can be determined by the solving the following equation:

\[
p_a + \frac{\alpha(N_a)\rho}{s} \tilde{N}_a = p_b + \alpha(\tilde{N}_a) T_Tb,
\]

and the number of transit commuters is \( \tilde{N}_b = N - \tilde{N}_a \). At equilibrium, the auto and transit modes will be indifferent to the \( \tilde{N}_a \) - th commuter, as Eq. (22) indicates. It can be verified that, at equilibrium, for the \( m \)-th commuter, \( \Delta(c(m)) = c_a(m) - c_b(m) \) would be negative for \( m < \tilde{N}_a \) and these commuters will choose the auto mode, and \( \Delta(c(m)) \) would be positive for \( m > \tilde{N}_a \) and these commuters will take transit. The auto commuting equilibrium is identical with that depicted in Fig. 1(a), where \( s \) is the bottleneck capacity and \( r = \frac{1}{\rho} \) is the departure rate from home for commuters. Note that this departure rate is also valid for the homogeneous case since \( \rho x = \beta \). At equilibrium, the travel cost of the \( m \)-th commuter is given by:

\[
c_{nc}(m) = \begin{cases} p_a + \alpha(m)\rho \frac{N_a}{s} & 0 \leq m \leq \tilde{N}_a \\ p_b + \alpha(m)T_Tb & N_a \leq m \leq N \end{cases}
\]

The subscript \( nc \) corresponds to the case where there is no parking space constraint.

**Lemma 1.** At the bi-modal equilibrium without the parking space constraint:

(i) \( c_{nc}(m) \) decreases in \( m \). If \( x \sim U(\bar{x}, \tilde{x}) \), the rate of decrease for \( m < \tilde{N}_a \) (auto commuters) is smaller than that for \( m > \tilde{N}_a \) (transit commuters).

(ii) \( c_{nc}(m) \) increases in \( x(m) \). For any VOT distribution, the rate of increase for \( m < \tilde{N}_a \) (auto commuters) is smaller than that for \( m > \tilde{N}_a \) (transit commuters).

**Proof.** Differentiating Eq. (23) with respect to \( m \) and \( x(m) \), we have

\[
\dot{c}_{nc}(m) = \begin{cases} \frac{\dot{x}(m)\rho}{s} \frac{N_a}{s} & 0 \leq m < \tilde{N}_a \\ \frac{\dot{x}(m)T_Tb}{s} & \tilde{N}_a \leq m \leq N \end{cases}
\]

Note that \( c_{nc}(m) \) is continuous but not differentiable at \( m = \tilde{N}_a \), and at \( x = \alpha(\tilde{N}_a) \). Since \( \dot{x}(m) < 0 \), we can readily see that \( \dot{c}_{nc}(m) < 0 \). Also note that it is assumed that \( p_a - p_b > 0 \). From Eq. (22), we know \( \frac{\dot{N}_b}{s} < T_Tb \). Since \( x \sim U(\bar{x}, \tilde{x}) \), then \( \dot{x}(m) = -\frac{1}{\beta^2}(\bar{x} - x) < 0 \), as a result, \( 0 < \dot{x}(m)\rho \frac{N_a}{s} < \dot{x}(m)T_Tb \).

3.1.2. Bi-modal equilibrium with parking space constraint

Now we turn to the bi-modal equilibrium when the total parking supply at CBD, i.e., \( M \), is unable to accommodate all the potential auto demand, i.e., \( \tilde{N}_a \). Since commuters with higher VOT are more willing to travel by auto to save his or her travel time cost, at the bi-modal equilibrium, the first \( M \)-th commuters with higher VOT will choose to travel by auto and the rest of commuters will take transit. Moreover, the auto and transit modes will be indifferent to the \( M \)-th commuter. We thus have the travel cost of the \( m \)-th commuter as follows:

\[
c_{np}(m) = \begin{cases} p_a + \alpha(m)\left(T_Tb - \frac{p_a - p_b}{\alpha(M)}\right) & 0 \leq m \leq M \\ p_b + \alpha(m)T_Tb & M \leq m \leq N \end{cases}
\]

The subscript \( np \) corresponds to the case without permit scheme introduced.

**Lemma 2.** At the bi-modal equilibrium with the parking space constraint,

(i) \( c_{np}(m) \) decreases in \( m \). If \( x \sim U(\bar{x}, \tilde{x}) \), the rate of decrease for \( m < M \) (auto commuters) is smaller than that for \( m > M \) (transit commuters).

(ii) \( c_{np}(m) \) increases in \( x(m) \). For any VOT distribution, the rate of increase for \( m < M \) (auto commuters) is smaller than that for \( m > M \) (transit commuters).

**Proof.** Similar to that for Lemma 1.
3.2. Equilibrium under non-expirable parking permit scheme

Given the number of permits distributed (identical with the number of parking reservations provided) per day, the equilibrium flow pattern under such a non-expirable parking permit scheme is similar to the one depicted in Fig. 2(a) for the homogeneous case. At equilibrium, the first \( M \)-th commuters with VOT ranging from \( \alpha \) to \( \alpha(M') \) will choose to reserve parking spaces via permits. The commuters with VOT ranging from \( \alpha(M') \) to \( \alpha(M) \) will choose to compete for unreserved parking spaces and travel by auto, and others will take transit. In addition, the \( M \)-th commuter will be indifferent between competing for parking and taking transit, the \( M' \)-th commuter will be indifferent between reserving a parking space via a permit and competing for parking.

The commuters with higher VOT are more willing to buy permits on the market to save their travel time and schedule delay cost. This is because, for auto commuters, by reserving a parking space, the reduction in the sum of travel time and travel-time-equivalent schedule delay, \( T_a(t) + \rho(t - t - T_d(t)) \) is identical. For the same reduction, commuters with higher VOT value it more and thus are more willing to buy permits. It is worth mentioning that, without transaction cost, the tradability of permits ensures the optimal allocation of permits among commuters with different VOT no matter what the initial permit distribution is. More specifically, the first \( M' \) commuters with higher VOT than others would always choose to acquire permits to reserve parking spaces and travel by auto.

At equilibrium, the permit price can be expressed as follows:

\[
p_{ne} = \frac{\alpha(M')}{\alpha(M)} \left( T_b - \frac{p_a - p_b - \frac{M'}{S}}{\alpha(M)} \right). \tag{26}
\]

Without considering the benefits from the permits distributed to a commuter, the travel cost of the \( m \)-th commuter can be expressed as:

\[
c_{ne}(m) = \begin{cases} 
    p_a + \alpha(m) \frac{M'}{S} + p_{ne} & 0 \leq m \leq M' \\
    p_a + \alpha(m) \left( T_b - \frac{p_a - p_b}{\alpha(M)} \right) & M' \leq m \leq M \\
    p_a + \alpha(m) T_b & M \leq m \leq N
\end{cases} \tag{27}
\]

where \( p_{ne} \) is the permit price determined by Eq. (26). The subscript \( ne \) corresponds to the case with the non-expirable permit scheme introduced.

**Lemma 3.** After introducing the non-expirable permit scheme,

(i) \( c_{ne}(m) \) decreases in \( m \). If \( \alpha \sim U[\alpha, \bar{\alpha}] \), the rate of decrease for \( m < M' \) (r-commuters) is smaller than that for \( M' < m < M \) (u-commuters), and the rate of decrease for \( M' < m < M \) (u-commuter) is smaller than that for \( m > M \) (transit commuters).

(ii) \( c_{ne}(m) \) increases in \( \alpha(m) \). For any VOT distribution, the rate of increase for \( m < M' \) (r-commuters) is smaller than that for \( M' < m < M \) (u-commuters), and the rate of increase for \( M' < m < M \) (u-commuters) is smaller than that for \( m > M \) (transit commuters).

**Proof.** Similar to that for Lemma 1. \( \square \)

In addition, the homogeneous case, we can derive the unique socially optimal number of permits (identical with the number of reservations) for distribution per day, which is given as follows:

\[
M^* = \begin{cases} 
    M & M \leq M' \\
    M & M < M'
\end{cases} \tag{28}
\]

where \( M \) solves:

\[
p_a + \frac{\alpha(M)}{S} \frac{M'}{S} + \int_0^\frac{M}{S} \frac{1}{\alpha(M)} dm = p_a + \frac{\alpha(M)}{S} \left( T_b - \frac{p_a - p_b}{\alpha(M)} \right) \tag{29}
\]

In Eq. (29), replacing \( M \) by \( M' \), we see that the left-hand side represents the marginal cost from adding an additional r-commuter while the right-hand side represents the marginal saving from reducing one u-commuter (or adding one r-commuter). Based on Eqs. (28) and (29), we have the following observation.

**Observation 3.** For the heterogeneous case, the optimal design of the non-expirable scheme requires commuters’ VOT information, besides the total parking supply and highway bottleneck capacity.

3.3. Equilibrium under expirable parking permit scheme

We propose the same permit scheme as that discussed in Section 2. In the scheme, there are \( k \) classes of permits and the \( i \)-th class of permits can be used for parking reservations with an expiration time \( t_i \), where \( i = 1, 2, \ldots, k \) and \( t_i < t_{i+1} \).
Similarly, for insufficient parking spaces $M$, by setting the ideal-scheme with $N = M$ and $k \rightarrow \infty$, the queue at the bottleneck and the competition for limited parking spaces can be completely eliminated. The equilibrium price of a parking permit with expiration time $t$ is given by
\[
p_e(t) = p_b - p_a + \alpha(M) \left( T_b - \rho \frac{M}{s} \right) + \rho \int_{t-\frac{s}{v}}^{t} \alpha((t' - w)s) dw,
\]
where $t \in [t - \frac{M}{v}, t']$. From Eq. (30), we see that commuters’ heterogeneity will influence the price of the expirable parking permit. However, it will not influence the performance of the scheme, i.e., congestion at the bottleneck and the deadweight loss of parking competition can still be fully eliminated. Differentiating Eq. (30) with respect to $t$, we have
\[
p_e'(t) = \rho \alpha((t' - t)s).
\]
Since $\alpha((t' - t)s) > 0$, Eq. (31) indicates the price of the expirable permit is an strictly increasing function of its expiration time, $t$. At equilibrium, the $m$-th commuter will choose to use the permit expires at $t = t' - \frac{m}{v}$ and arrive at the parking space at that expiration time. Among the $M$ auto commuters, to see the $m$-th commuter has no incentive to use another permit and depart at another time, we examine the following cost function for the $m$-th commuter to depart at $t$ under the socially optimal flow pattern:
\[
c_e(m, t) = p_a + \alpha(m) \rho(t' - t) + p_e(t).
\]
From Eq. (32), we further have the following:
\[
\frac{\partial c_e(m, t)}{\partial t} = -\rho \alpha(m) + \rho \alpha((t' - t)s); \quad \frac{\partial^2 c_e(m, t)}{\partial t^2} = -\rho \alpha((t' - t)s)s.
\]
Note that when $t = t' - \frac{m}{v}$, from Eq. (33), we have
\[
\frac{\partial c_e(m, t)}{\partial t} = 0; \quad \frac{\partial^2 c_e(m, t)}{\partial t^2} = -\rho \alpha(m)s > 0.
\]
Thus for the $m$-th commuter, departing at $t = t' - \frac{m}{v}$ minimizes his or her travel cost. In addition, the equilibrium travel cost of all commuters can be expressed as follows:
\[
c_e(m) = \begin{cases} 
  p_a + \alpha(m) \rho \frac{m}{v} + p_b \left( t' - \frac{m}{v} \right) & 0 \leq m \leq M \\
  p_b + \alpha(m) T_b & M \leq m \leq N 
\end{cases}
\]
The subscript $e$ corresponds to the case with the ideal-scheme introduced.

**Lemma 4.** After introducing the ideal-scheme,

(i) $c_e(m)$ decreases in $m$. If $\alpha \sim U[\alpha, \overline{\alpha}]$, the rate of decrease for $m < M$ (r-commuters) is smaller than that for $M < m \leq N$ (transit commuters).

(ii) $c_e(m)$ increases in $\alpha(m)$. For any VOT distribution, the rate of increase for $m < M$ (r-commuters) is smaller than that for $M < m \leq N$ (transit commuters).

**Proof.** Similar to that for Lemma 1. □

We summarize Lemmas 1–4 in the following proposition.

**Proposition 3.** For all cases (with or without the parking space constraint and under non-expirable or expirable permit scheme),

(i) Equilibrium travel cost $c(m)$ decreases in $m$. If $\alpha \sim U[\alpha, \overline{\alpha}]$, the rate of decrease is the smallest for $r$-commuters, followed by $u$-commuters and then transit commuters.

(ii) Equilibrium travel cost $c(m)$ increases in $\alpha(m)$. For any VOT distribution, the rate of increase is the smallest for $r$-commuters, followed by $u$-commuters and then transit commuters.

**Observation 4.** For the heterogeneous case, the optimal design of the ideal-scheme does not require commuters’ VOT information, and is robust to the variation in commuters’ VOT.

Observation 4 is an extension of Observation 2 to the heterogeneous case. We now show in the heterogeneous case, under the scheme with $k$ steps (equally divided), the loss of efficiency in total social cost reduction can also be bounded.
Similar to the homogeneous case, we consider the total social cost (auto mode only) under the scheme with $M' = M$ and $k$ steps, which can be expressed as follows:

$$TC_{he}(k) = \sum_{m=1}^{k} \int_{(i-1)}^{i} p_a + \frac{x(m)\rho_i M}{k} \frac{dm}{s}.$$  \hspace{1cm} (35)

The subscript $he$ represents the case with heterogeneous commuters. Similar to Eq. (9), we then define the efficiency loss measure for the heterogeneous case:

$$l_{he}(k) = \frac{TC_{he}(k) - TC_{he}(\infty)}{TC_{he}(\infty)}.$$  \hspace{1cm} (36)

**Proposition 4.** For the heterogeneous case, the efficiency loss

$$l_{he}(k) < \frac{H_k}{k},$$  \hspace{1cm} (37)

where $H_k = \sum_{i=1}^{k} (1/i)$ is the $k$ - th harmonic number.

**Proof.** See Appendix A. □

From Proposition 1, we know that in the homogenous case, a permit scheme with 10 steps would only lose efficiency less than 10% when compared with the ideal-scheme. Proposition 4 then suggests that in the heterogeneous case, for a general VOT distribution, such a scheme would lead to an efficiency loss less than 30%. Similarly, we will show in the following, if setting $\lambda < 1$ would lead to efficiency loss, the loss can also be bounded

Firstly, we express the total social cost under the optimal scheme with $k$ steps and given $M'$ as follows:

$$TC_{he}(k, M') = \sum_{i=1}^{k} \int_{(i-1)}^{i} p_a + \frac{x(m)\rho_i M}{k} \frac{dm}{s} + \sum_{i=1}^{k} \int_{(i-1)}^{i} p_a + \frac{x(m)(p_b + \frac{x(M)T_b - p_a}{x(M)})}{s} \frac{dm}{s},$$  \hspace{1cm} (38)

where $k \geq 1$. The subscript $he$ represents the case with heterogeneous commuters. For given $k$, if the scheme with $M' = \lambda M$ would lead to efficiency loss, similar to Eq. (13), we define the percentage efficiency loss measure:

$$l_{he}(\lambda) = \frac{TC_{he}(k, \lambda M) - TC_{he}(k, M)}{TC_{he}(he, M)}.$$  \hspace{1cm} (39)

**Proposition 5.** For the heterogeneous case, the efficiency loss $l_{he}(\lambda) < 1 - \lambda$.

**Proof.** See Appendix B. □

According to Proposition 5, a permit scheme with $k$ steps and $M' = 0.9M$ would only lose efficiency less than 10% when compared with the scheme with $k$ steps and $M' = M$. Similar with the homogenous case, we denote the efficiency loss of the scheme with $k$ steps and $M' = \lambda M$ by $\epsilon_{he}(k, \lambda)$. From Propositions 4 and 5, we know that $\epsilon_{he}(k, \lambda)$ is less than $1 - \lambda \left(1 - \frac{H_k}{k}\right)$. Therefore, a scheme with 10 steps and 90% reserved parking spaces would yield a loss less than $1 - (1 - 30\%)(1 - 10\%) = 37\%$.

3.4. Welfare effects under different permit distributions

As already mentioned in Section 2.4, the initial permit distribution will not influence the commuting equilibrium due to the tradability of the parking permits assuming no transaction cost. This is also true when user heterogeneity is considered. However, the initial permit distribution will affect the welfare distribution among commuters. Based on the results in Sections 3.1-3.3, we will now discuss the welfare effects of different permit distributions under non-expirable permit scheme and the ideal-scheme. For the non-expirable parking permit scheme, suppose that the number of permits distributed to the $m$-th commuter is $n_m$ per day, and $P_{ne} = n_mP_{ne}$ would be the economic benefit from selling out all permits one has, where $P_{ne}$ is defined by Eq. (26); and for the ideal-scheme, suppose the number of permits with expiration time $t$ distributed to the $m$-th commuter is $n_m(t)$ per day, then we have $P_{ne} = \int_{t}^{e} n_m(t)p_e(t)dt$, and $p_e(t)$ is determined by Eq. (30). From Eqs. (27) and (34), we can readily obtain the travel cost of individual commuters after taking into account the benefit from permits distributed, which is expressed as follows:

$$c_i(m, P_m) = c_i(m) - P_m,$$  \hspace{1cm} (40)

where $i = ne$ or $e$, in correspondence to the non-expirable scheme and the ideal-scheme respectively. In the following analysis, we firstly present the uniform permit distribution as a benchmark, and then compare it to permit distributions with equal cost-reduction.
3.4.1. Uniform permit distribution

If we measure commuters’ welfare in monetary units, the uniform permit distribution provides everyone the same amount of money, i.e., \( P_m \) is identical for every commuter. However, this distribution may cause inequity among commuters since people with higher VOT may receive more benefit from the permit charging scheme. This will be shown later in Fig. 3, where the black solid lines correspond to the individual travel cost when all commuters are competing for parking spaces, i.e., \( c_{np}(m) \) defined in Eq. (25), the black dotted lines correspond to individual travel costs when benefits from selling permits is not included, i.e., \( c_{i}(m) \) where \( i = ne \) or \( e \); and the blue solid lines represent individual travel cost when the benefits from permits is included, i.e., \( c_{i}(m, P_m) \) where \( i = ne \) or \( e \).

Fig. 3(a) and (b) respectively depict the individual cost change after the non-expirable permit scheme and the ideal-scheme is introduced. In both cases, commuters using permits to reserve parking spaces have higher VOT than transit commuters, and the cost reduction decreases with their orders and thus increases with their VOT. Because the uniform permit distribution provides all commuters the same amount of benefits, those using permits actually benefit more from the permit scheme.

3.4.2. Equal cost-reduction distribution

Considering the commuters with higher VOT may benefit more under the permit scheme with uniform distribution, we now propose a distribution that leads to equal cost-reduction for everyone. Firstly we look at the case with non-expirable permits. The total economic benefit from such a scheme is given as follows:

\[
TB_{ne} = \int_0^N c_{np}(m) - c_{ne}(m, P_m) dm, \tag{41}
\]

In order to make everyone has the same cost reduction, the number of permits should be distributed to the \( m \)-th commuter is:

\[
n_m = \begin{cases} 
\frac{1}{P_{ne} N} TB_{ne} - \frac{c_{np}(m) - c_{ne}(m)}{P_{ne}} & 0 \leq m \leq M' \\
\frac{1}{P_{ne} N} TB_{ne} & M' \leq m \leq N 
\end{cases}, \tag{42}
\]

where \( c_{ne}(m) \) is given by Eq. (27). Further, taking the first-order derivative of Eq. (42) yields:

\[
\dot{n}_m = \begin{cases} 
- \frac{\dot{c}_{np}(m)}{P_{ne}} \left( T_b - \frac{P_{ne} - P_{be}}{\dot{c}_{np}(m)} \right) & 0 \leq m < M' \\
0 & M' < m \leq N 
\end{cases}. \tag{43}
\]

Note that, when \( 0 \leq m < M' \), \( \dot{n}_m > 0 \) holds, which implies that fewer permits should be given to \( r \)-commuters with higher VOT. Also note that all the \( u \)-commuters and transit commuters should receive the same number of permits. To summarize, we have the following proposition:

---

**Fig. 3.** Welfare effect of uniform permit distribution.
Proposition 6. Under the non-expirable scheme, the equal cost-reduction permit distribution would allocate fewer permits to r-commuters with higher VOT. All the u-commuters and transit commuters should receive the same number of permits, which is more than that received by r-commuters.

Similarly, we can derive the total economic benefits from the ideal-scheme:

\[ TB_e = \int_0^N c_{np}(m) - c_e(m, P_m) \, dm. \]  

(44)

In order to make everyone experience the same cost reduction, the total value of the permits distributed to the \( m \)-th commuter is:

\[ P_m = \begin{cases} \frac{T_B}{N} - (c_{np}(m) - c_e(m)) & 0 \leq m \leq M \\ \frac{T_B}{N} & M \leq m \leq N \end{cases} \]  

(45)

where \( c_e(m) \) is given by Eq. (34). Similar to the homogeneous case, here only the total value of permits matters, commuters can choose to receive either fewer permits with a higher price or more permits but with a lower price. Taking the first-order derivative of Eq. (45) yields,

\[ \dot{P}_m = \begin{cases} -2(m)(T_B - \frac{P_r P_b}{2 \lambda M}) - \rho \frac{m}{M} & 0 \leq m < M \\ 0 & M \leq m \leq N \end{cases} \]  

(46)

When \( 0 \leq m < M, \dot{P}_m > 0 \) holds, which implies r-commuters with higher VOT should receive fewer permits and all transit commuters should receive the same total value of permits, which is larger than that received by r-commuters.

Proposition 7. Under the ideal-scheme, the equal cost-reduction permit distribution would allocate lower value of permits to r-commuter with higher VOT and an equal value of permits to transit commuters. The value of permits received by transit commuters would be higher than that received by r-commuters.

According to the results above, we draw Fig. 4(a) and (b) to depict the welfare effects of this new permit distribution under non-expirable permit scheme and the ideal-scheme respectively. In Fig. 4, the newly added blue real lines correspond to the travel cost of the new equal cost-reduction distribution. As shown in Fig. 4, compared to a uniform permit distribution, it allocate fewer permits to some r-commuters with higher VOT and more permits to the other r-commuters with lower VOT and the remaining commuters.

4. Comparison with pricing scheme

In this section, we first discuss optimal time-varying tolls to eliminate deadweight loss of parking competition and bottleneck congestion for the cases with homogeneous and heterogeneous commuters, respectively. Then, we will compare it with the ideal-scheme where \( M' = M \) and \( k \to \infty \).

---

**Fig. 4.** Welfare effect of equal cost-reduction permit distribution.
4.1. Optimal pricing scheme

4.1.1. Homogeneous commuters

Under the optimal pricing scheme, parking spaces are fully utilized, and high schedule delay cost due to competition for parking is reduced to the minimum and congestion at the bottleneck is eliminated. At equilibrium, the M-th commuter should still be indifferent between auto and transit mode. The time-varying toll can be determined as follows:

\[ \tau(t) = \begin{cases} p_b - p_a + \tau(M) (T_b - \rho \frac{M}{T_b}) & t < t' - \frac{M}{T_b} \\ p_b - p_a + \tau(M) (T_b - \rho \frac{t}{T_b}) + \rho \int_{t'}^{t'} \tau((t - w)s)dw & t' - \frac{M}{T_b} \leq t \leq t' \end{cases} \] (47)

Indeed, since there will be no departure before \( t' - \frac{M}{T_b} \), the socially optimal toll before that time cannot be uniquely determined. It only need be large enough so that all commuters have no incentive to depart earlier. Thus, Eq. (47) only presents one alternative. However, the toll between \( t' - \frac{M}{T_b} \) and \( t' \) is uniquely determined. After this scheme is imposed, every commuter’s travel cost will be the same as the travel cost of taking transit. However, the queue at the bottleneck and deadweight loss of competition for parking is fully eliminated.

4.1.2. Heterogeneous commuters

The optimal pricing scheme should yield the same departure pattern as above, i.e., all parking spaces are used and the schedule delay cost is reduced to minimum, and congestion at the bottleneck is eliminated. At equilibrium, the M-th commuter will also be indifferent between the auto and transit mode. The time-varying toll can be determined as follows:

\[ \tau(t) = \begin{cases} p_b - p_a + \tau(M) (T_b - \rho \frac{M}{T_b}) & t < t' - \frac{M}{T_b} \\ p_b - p_a + \tau(M) (T_b - \rho \frac{t}{T_b}) + \rho \int_{t'}^{t'} \tau((t - w)s)dw & t' - \frac{M}{T_b} \leq t \leq t' \end{cases} \] (48)

Again, the above toll pattern before \( t' - \frac{M}{T_b} \) is one of many alternative. However, the toll between \( t' - \frac{M}{T_b} \) and \( t' \) is unique. Among M auto commuters, the m-th commuter will depart at \( t = t' - \frac{M}{T_b} \), and has no incentive to depart at another time. To see this, considering the travel cost for the m-th commuter under the socially optimal flow pattern:

\[ c_{pr}(m, t) = p_a + \tau(m) \rho(t' - t) + \tau(t), \] (49)

where \( \tau(t) \) is given by Eq. (48). Differentiating Eq. (49) yields:

\[ \frac{\partial c_{pr}(m, t)}{\partial t} = -\rho \tau(m) + \rho \tau((t' - t)s); \quad \frac{\partial^2 c_{pr}(m, t)}{\partial t^2} = -\rho \tau((t' - t)s). \] (50)

Note that when \( t = t' - \frac{M}{T_b} \), from Eq. (50), we have

\[ \frac{\partial c_{pr}(m, t)}{\partial t} = 0; \quad \frac{\partial^2 c_{pr}(m, t)}{\partial t^2} = -\rho \tau(m)s > 0. \]

Thus for the m-th commuter, departing at \( t = t' - \frac{M}{T_b} \) minimizes his or her travel cost.

4.2. Pricing vs. tradable expirable permits

As we show above, an optimal pricing scheme can eliminate and the queue at the bottleneck and minimize travel cost of commuters. However, the design of such a scheme requires complete information of commuters’ VOT as shown in Eqs. (47) and (48). Otherwise, a trial-and-error procedure will be needed for the implementation of the scheme. (e.g., Li, 2002; Yang et al., 2004; Han and Yang, 2009; Wang and Yang, 2012; Wang et al., 2013).

In comparison, the proposed ideal-scheme can achieve the same performance, while its design and performance is irrelevant to commuters’ VOT distribution. Commuters’ VOT will only influence the permit price and does not affect the overall performance of the scheme. It is certainly impractical for the system planner to set such an ideal-scheme with an infinite number of steps (note that it is impractical to set a time-varying pricing scheme proposed above either). However, as we have showed in Propositions 1 and 4, the efficiency loss of the permit scheme with finite steps can be bounded. For the homogeneous case, a scheme with ten steps will suffer an efficiency loss less than 10%; while for the heterogeneous case, the efficiency loss is less than 30%. In addition, Propositions 2 and 5 suggests that the efficiency loss due to setting an undesirable \( M' \) can also be bounded. This provides additional flexibility for the parking management authority in implementing the scheme.

Besides the above advantage, the proposed permit scheme inherits the characteristics of tradable mobility credits in the sense that it achieves the control objective with minimum user costs. Secondly, it is revenue neutral and may thus face less public objection. Thirdly, when justified, the permit distribution can be designed to improve equity among commuters.
5. Conclusion and discussion

5.1. Conclusion

This study has developed a novel tradable parking permit scheme to realize parking reservations of commuters when they are either homogeneous or heterogeneous in their VOT. It is found that the ideal-scheme with an infinite number of steps is superior to the time-varying pricing scheme in the sense that it needs no information on commuters’ VOT and its performance is robust to commuters’ VOT variation.

Although it is unlikely to implement a permit scheme with an infinite number of steps, the efficiency loss of a scheme with finite steps can be bounded in both cases with homogeneous and heterogeneous commuters. More specifically, for the former, the efficiency loss is strictly less than the reciprocal of the number of steps, while for the latter the efficiency loss is strictly less than the reciprocal of the harmonic mean of the integers from one to the number of steps. This implies for a scheme with, e.g., ten steps, the efficiency loss will be less than 10% for the homogenous case and 30% for the heterogeneous case. Furthermore, the efficiency loss due to non-optimal setting of the number of reserved parking spaces can also be bounded.

When commuters are heterogeneous, the permit scheme might cause unfairness among commuters different VOT. Thus, we have introduced an equal cost-reduction permit distribution. With non-expirable permits, such a distribution would allocate fewer permits to r-commuters with higher VOT. Further, all u-commuters and transit commuters should receive the same number of permits, which is more than what r-commuters receive. Under the ideal-scheme, the distribution would assign lower value of permits to r-commuters with higher VOT, and give all transit commuters equal value of permits, which is more than that given to r-commuters. However, we would like to point out that the two permit distributions, i.e., uniform permit distribution and equal cost-reduction distribution, are proposed and discussed to provide insight regarding the equity issue. It might not be reasonable to allocate fewer permits to commuters with higher VOT, and the equal cost-reduction distribution might be unfair to those commuters.

5.2. Discussions

In this study, we assume no late arrivals. If late arrivals are allowed, the expirable parking permit schemes cannot completely eliminate the queue at the bottleneck (Liu et al., 2014). This is because late r-commuters have incentive (to reduce late arrival penalty) to arrive at destination closer to $t^*$. This incentive would lead them to form a queue at the bottleneck. Another issue when late arrivals are incorporated is that the optimal design of the permit scheme will reply on commuters’ VOT information. Otherwise, the socially optimal departure time interval would not be known. However, the design of the scheme is still less information-demanding than the pricing scheme. Once we know the socially preferable departure time interval of commuters, the design of the permit scheme needs no more information on VOT. In contrast, the design of an optimal pricing scheme will still need the complete information of VOT.

Further research may consider incorporating the parking cruising time. As mentioned in Liu et al., 2014, parking cruising time can be incorporated into the current model by introducing a parking cruising time function, which decreases with respect to the number of remaining parking spaces. Since the central authority can assign a specific parking space to a commuter after he or she makes a reservation through a permit (location information of the parking space will be sent to this commuter also), commuters need no longer to cruise for their reserved parking spaces. Thus, the proposed scheme has the potential to reduce deadweight loss due to cruising.

This study focuses on the impact of the parking spaces constraint on the bi-modal traffic equilibrium. For simplicity, all parking spaces are inexplicitly assumed to have identical features. For further research, it would be interesting to take into account differentiated parking, i.e., the considered parking spaces may have different parking fees, and different distances to the destination. In this case, the commuters’ incentive or willingness to compete for parking spaces of different types would be different, which is a reflection of commuters’ different valuations on the differentiated parking spaces. Besides, parking fees might be utilized as an instrument for managing traffic.

The current study considers that commuters have different values of time and schedule delay penalties. However, the ratio of early arrival penalty to value of time, i.e., $p = \beta/\alpha$, is assumed to be constant and identical for all commuters. Further study might consider different types of commuter heterogeneity. For instance, commuters may have different $p = \beta/\alpha$ (a simple example could be that all commuters have identical schedule delay penalty, but their VOT varies). For different types of user heterogeneity, the modeling framework in the paper can still be applied. The ideal-scheme discussed here would still not rely on commuters’ VOT information. However, the resulting equilibrium permit price, as well as the welfare effects of the permit scheme, can be different since they depend on commuters’ VOT distribution. In addition, when we consider a more practical scheme with finite steps, r-commuters within a step might have different preferences on departure times depending on their $p$. The efficiency loss of the scheme with finite steps can also be different, which depends on commuters’ VOT distribution.

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Appendix A. Proof of Proposition 4

Proof. From Eqs. (35) and (36), we have the following:

\[ l_n(k) = \sum_{i=1}^{k} \int_{0}^{\frac{m}{k}} y_n(m) dm - \int_{0}^{\frac{m}{k}} y_{n}(m) \rho \left( \frac{M}{y} \right) dm = \sum_{i=1}^{k} \int_{0}^{\frac{m}{k}} \frac{\rho}{2} \left( \frac{M}{y} - \frac{m}{k} \right) dm. \]  

(51)

From Eq. (22), the parking constraint, \( M < N_a \) implies that:

\[ p_a + \alpha(M) \rho \frac{M}{s} < p_b + \alpha(M) T_b \Rightarrow \frac{p_b}{\alpha(M)} + \alpha(M) T_b - p_a > \rho \frac{M}{s}. \]  

(52)

Then, from Eqs. (51) and (52), we have the following:

\[ l_n(k) = \sum_{i=1}^{k} \int_{0}^{\frac{m}{k}} \alpha(m) \rho \left( \frac{M}{y} \right) dm \left\{ \frac{\rho}{2} \left( \frac{M}{y} - \frac{m}{k} \right) \right\} \]  

\[ < \sum_{i=1}^{k} \int_{0}^{\frac{m}{k}} \alpha(m) \rho \left( \frac{M}{y} \right) dm \left\{ \frac{\rho}{2} \left( \frac{M}{y} - \frac{m}{k} \right) \right\}. \]  

(53)

For \( (i-1) \frac{m}{k} \leq m < i \frac{m}{k} \) let:

\[ x_i(m) = \alpha(m) \rho \left( \frac{M}{y} - \frac{m}{k} \right); \quad y_i(m) = \alpha(m) \rho \left( \frac{M}{y} - \frac{m}{k} \right). \]  

(54)

From Eq. (54), we know that:

\[ \frac{x_i(m)}{y_i(m)} = \frac{\alpha(m) \rho \left( \frac{M}{y} - \frac{m}{k} \right)}{\alpha(m) \rho \left( \frac{M}{y} - \frac{m}{k} \right) < \frac{\alpha(m) \rho \left( \frac{M}{y} - \frac{i-1}{k} \frac{m}{y} \right)}{\alpha(m) \rho \left( \frac{M}{y} - \frac{i-1}{k} \frac{m}{y} \right) = \frac{1}{k+1-i}.} \]  

(55)

From Eqs. (53) and (55), we have:

\[ l_n(k) < \sum_{i=1}^{k} \frac{\alpha(m) \rho \left( \frac{M}{y} - \frac{m}{k} \right) dm}{\sum_{i=1}^{k} y_i(m) dm} \]  

\[ = \frac{\sum_{i=1}^{k} \frac{1}{k+1-i} \frac{\alpha(m) \rho \left( \frac{M}{y} - \frac{m}{k} \right) dm}{\sum_{i=1}^{k} y_i(m) dm}. \]  

(56)

From Eq. (54), we know that:

\[ y_i(m) = \alpha(m) \rho \left( \frac{M}{y} - \frac{m}{k} \right) + \alpha(m) \rho \left( \frac{1}{y} \right) < 0. \]  

(57)

Thus for \( 1 \leq g < h \leq k, \)

\[ \int_{(g-1)}^{y_h(m)} \frac{\alpha(m) \rho \left( \frac{M}{y} - \frac{m}{k} \right) dm}{\sum_{i=1}^{k} y_i(m) dm}. \]  

(58)

Also note for \( 1 \leq g < h \leq k, \)

\[ \frac{1}{k+1-g} < \frac{1}{k+1-h}. \]  

(59)

Then, from Eqs. (56), (58), and (59), we have:

\[ l_n(k) < \sum_{i=1}^{k} \frac{1}{k+1-i} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{k+1-i} \frac{H_k}{k}. \]  

(60)

This completes the proof. □

Appendix B. Proof of Proposition 5

Proof. Firstly, note that the parking space constraint implies that

\[ M < N_a = \frac{s}{\rho} \left( p_b + \alpha(M) T_b - p_a \right) \Rightarrow p_b + \alpha(M) T_b > p_a + \alpha(M) \rho \frac{M}{s}. \]  

(61)

From Eqs. (25), (35), and (38), it can be verified that:
\[
\hat{k}_n(\lambda) = \sum_{i=1}^{k} \int_{h(i-1)}^{h(i)} \lambda (m) \left( \frac{b_0 + a(M) T_b - a}{a(M)} - \lambda \frac{s}{M} \right) dm.
\]  

(62)

For \((i-1) \frac{h}{s} \leq m < i \frac{h}{s}\), let:

\[
z(m) = \lambda (m) \left( \frac{b_0 + a(M) T_b - a}{a(M)} - \lambda \frac{s}{M} \right).
\]

(63)

Then, Eq. (62) can be written as:

\[
\hat{k}_n(\lambda) = \sum_{i=1}^{k} \int_{h(i-1)}^{h(i)} \lambda (m) dm.
\]

(64)

For \(1 \leq g < h \leq k\),

\[
\alpha_x(m) > \alpha_y(m) \text{and} -\rho \frac{g M}{s} > -\rho \frac{h M}{s}.
\]

(65)

From Eqs. (61), (63) and (65), we know that for \(1 \leq g < h \leq k\),

\[
z_x(m) > z_y(m).
\]

(66)

Therefore, for \(1 \leq g < h \leq k\),

\[
\int_{(g-1)}^{(h-1)} z_x(m) dm > \int_{(g-1)}^{(h-1)} z_y(m) dm.
\]

(67)

From Eqs. (67) and (64), we know that \(\hat{k}_n(\lambda) < (1 - \lambda)\) for \(\lambda < 1\).

References


