Expansible parking reservations for managing morning commute with parking space constraints

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Abstract

When total parking supply in an urban downtown area is insufficient, morning commuters would choose their departure times not only to trade off bottleneck congestion and schedule delays, but also to secure a parking space. Recent studies found that an appropriate combination of reserved and unreserved parking spaces can spread the departures of those morning commuters and hence reduce their total travel cost. To further mitigate both traffic congestion and social cost from competition for parking, this study considers a parking reservation scheme with expiration times, where commuters with a parking reservation have to arrive at parking spaces for the reservation before a predetermined expiration time. We first show that if all parking reservations have the same expiration time, it is socially preferable to set the reservations to be non-expirable, i.e., without expiration time. However, if differentiated expiration times are properly designed, the total travel cost can be further reduced as compared with the reservation scheme without expiration time, since the peak will be further smoothed out. We explore socially desirable equilibrium flow patterns under the parking reservation scheme with differentiated expiration times. Finally, efficiencies of the reservation schemes are examined.

1. Introduction

Parking is a growing problem in many downtown areas around the world. The time spent on searching for a parking space often constitutes a substantial portion of travel cost of individual drivers, and thus has considerable impacts on their travel choices. Bifulco (1993) introduced parking search times, types and fees in a static stochastic traffic assignment model for the evaluation of various parking policies. Arnott and Rowse (1999) examined the steady-state equilibria of cars cruising for parking on a circle when parking is unsaturated. Arnott and his collaborators have conducted a series of follow-up studies on the parking problem integrating traffic congestion and on-street or curbside parking (e.g., Arnott and Inci, 2006, 2010; Arnott and Rowse, 2009).

Besides parking search time, parking fee is another factor that may significantly affect commuters’ travel behaviors. Many researchers have considered parking fee as an instrument to help manage traffic. In a static context, Glazer and Niskanen (1992) optimized parking fees and showed that if road usage is suboptimally priced, a lump-sum parking fee can increase social welfare. Arnott et al. (1991) concluded that parking fees alone can be efficient in increasing social welfare in morning commute and a combination of road tolls and parking fees can yield system optimum. Verhoef et al. (1995) conducted a diagrammatic analysis on how parking fees and physical restrictions on parking space supply affect individual travel cost.
and modal split. Anderson and de Palma (2004) treated parking as a common property resource and examine the benefits of pricing it. Zhang et al. (2008) derived the daily commuting pattern that combines both the morning and evening commutes, and investigated mechanisms and efficiencies of several road toll and parking fee regimes. To account for the temporal aspect of parking, Zhang and van Wee (2011) further introduced a duration-dependent parking fee scheme into their daily commuting model. More recently, Qian et al. (2012) investigated how parking fee and parking supply can be optimized to mitigate traffic congestion. Fosgerau and de Palma (2013) proposed the parking fee at the workplace as a substitute of commuting model. More recently, Qian et al. (2012) investigated how parking fee and parking supply can be optimized to mitigate traffic congestion.

In addition to parking fees and search times, previous studies also considered how the availability of parking shapes commuters’ travel patterns. It is reported in Qian et al. (2011) that parking availability will affect commuters’ choices of departure time, travel mode and route. Zhang et al. (2011) explored the morning commuting equilibrium when the destination provides inadequate parking spaces to accommodate potential private cars. In the spirit of tradable mobility credits proposed by Yang and Wang (2011), Zhang et al. (2011) introduced a tradable parking permit scheme for managing the morning commute with limited downtown parking spaces. Recently, Yang et al. (2013) proved that, when the parking supply is insufficient, dedicating some spaces for reservations while keeping the rest open for competition can smooth out traffic arrivals to the bottleneck and thus reduce the total system cost. This is because commuters without a reserved parking space are compelled to leave home earlier to secure a public parking space.

In Yang et al. (2013), reservations of a parking space do not expire. They are valid no matter when commuters with a reservation choose to arrive. This study considers an expirable parking reservation scheme under which commuters with a parking reservation have to arrive at the parking spaces for the reservation before a predetermined expiration time. We first consider the simplest situation when all parking reservations have the same expiration time. As such, the parking reservation scheme considered in Yang et al. (2013) is indeed a special case of our scheme where the identical expiration time is sufficiently large. We then consider parking reservations with different expiration times. In this case, total travel cost may be further reduced since departures of morning commuters will be more dispersed due to differentiated expiration times. In other words, their departures are staged by those different expiration times. Finally, we examine the efficiencies of parking reservation schemes with identical and differentiated expiration times.

Following Yang et al. (2013), we consider a traffic corridor with two travel modes: a highway and a parallel transit line with dedicated right-of-way. For the highway, a single bottleneck is considered in order to make our model more tractable. Vickrey (1969) was the first to introduce the bottleneck model of congestion dynamics. Existence and uniqueness of the time-dependent equilibrium distribution of arrivals at a single bottleneck have been established in Smith (1984) and Daganzo (1985) respectively. The bottleneck model can generate insights concerning the formation and mitigation of traffic congestion. It has been extended to study various issues, including congestion pricing (Arnott et al., 1990; Laif, 1994; Xiao et al., 2012), Pareto-improving strategies (Daganzo and Garcia, 2000; Xiao and Zhang, 2013), demand elasticity (Arnott et al., 1993a; Yang and Huang, 1997), heterogeneous commuters (Newell, 1987; Arnott et al., 1994; Lindsey, 2004; van den Berg and Verhoef, 2011; Liu and Nie, 2011; Doan et al., 2011), small networks including routes in parallel and routes in series (Kuwahara, 1990; Arnott et al., 1993b; Tabuchi, 1993; Zhang and Zhang, 2010), pricing on general queuing networks (Yang and Meng, 1998), stochastic capacity and demand (Arnott et al., 1999; Lindsey, 2009; Xiao et al., 2013b), integration of both morning and evening peak hour commute (Zhang et al., 2005; Gonzales and Daganzo, 2013; Daganzo, 2013), travel time reliability (Yin et al., 2004; Li et al., 2012), congestion derivatives (Yao et al., 2010; 2012), tradable mobility credits (Tian et al., 2013; Nie and Yin, 2013; Xiao et al., 2013a).

The remainder of the paper is organized as follows. Section 2 revisits the Wardropian bi-modal equilibrium under insufficient parking spaces and with parking reservations. In Sections 3 and 4, parking reservation schemes with identical and differentiated expiration times will be discussed, respectively. Section 5 evaluates the efficiencies of expirable parking reservation schemes. Finally, Section 6 concludes the paper and provides some discussions.

2. Basic setting and considerations

It is assumed that there are N commuters traveling from a residential area to the Central Business District (CBD) every morning with two travel modes to choose: a transit line with dedicated right-of-way and a parallel highway with a single bottleneck. Commuters are assumed to have a common preferred arrival time t at the destination, and early or late arrival will be penalized. It is also assumed that parking spaces are located at the destination, and the walking time between parking spaces and the workplace is ignored for simplicity. The numbers of auto and transit commuters are denoted by N^A and N^T respectively, and N = N^A + N^T.

2.1. Generalized travel cost

2.1.1. Auto commuters

The auto commuter’ travel cost consists of his or her actual travel time cost and early or late arrival penalty. Departing at time t, the travel cost of the auto commuter is given by
\[
c(t) = \alpha \cdot T(t) + \beta \cdot \max\{0, t^* - t - T(t)\} + \gamma \cdot \max\{0, t + T(t) - t^*\}
\]
where \(T(t)\) is the travel time, \(\alpha, \beta, \gamma\) are the value of travel time, the marginal cost of early arrival and late arrival respectively. It is assumed that \(\gamma > \alpha > \beta > 0\). Also, in the following analysis, we employ the notation: \(\delta = \beta \gamma (\beta + \gamma)\). Without loss of generality, the free-flow commute time is assumed to be zero, thus \(T(t)\) only contains the queuing time, i.e., \(T(t) = q(t)/s\), where \(q(t)\) is the queue length at the highway bottleneck at time \(t\), and \(s\) is the service capacity of the bottleneck. The auto commuters experience no delay until the traffic flow exceeds the capacity. Once it happens, a deterministic queue will develop. We thus have:

\[
\frac{dq(t)}{dt} = \begin{cases} r(t) - s \cdot r(t) > s \text{ or } q(t) > 0 \\ 0 \cdot r(t) < s \text{ and } q(t) = 0 \end{cases}
\]

where \(r(t)\) is the flow rate arriving at the bottleneck at time \(t\). With this common setting in the literature, the travel cost of an individual auto commuter at the departure time equilibrium is

\[
P^A(N^A) = \delta \frac{N^A}{s}
\]

Clearly, the travel cost is a strictly increasing linear function of the number of auto commuters.

2.1.2. Transit commuters

Since we want to focus on the influence of the parking space constraint and auto mode, for simplicity, we consider the travel cost of riding transit as a strictly increasing function of the number of transit commuters, i.e.,

\[
P^T = P^T(N^T)
\]

We would like to point out that Eq. (4) is a reduced form of transit cost function for a model in which transit users also have a time-of-use decision and are subject to schedule delay costs (e.g., Kraus and Yoshida, 2002; Kraus, 2003).

2.2. Bi-modal equilibrium

2.2.1. Bi-modal equilibrium without parking space constraint

Assume \(P^A(N) > P^A(0)\) and \(P^A(N) > P^A(1)\), with Eqs. (3) and (4), we immediately have an interior equilibrium at which both auto and transit are used for commuting. In this case, the numbers of auto and transit commuters, \(N^A\) and \(N^T\) are determined by \(P^A(N^A) = P^A(N^T)\) and \(N^A + N^T = N\). This bi-modal equilibrium implies a perfect substitution between road and transit, leading to what is termed in the literature as a deterministic Wardropian bi-modal equilibrium.

2.2.2. Bi-modal equilibrium with parking space constraint

We now revisit the bi-modal equilibrium when the total parking supply at CBD, denoted as \(M\), is unable to accommodate all the potential auto demand, i.e., \(N^A\). Since the parking supply is insufficient, all commuters have to compete for them on a first-come-first-served basis. The competition will force commuters of auto mode to depart earlier such that at the bi-modal equilibrium, the travel cost of auto commuters will be identical to that of transit commuters, \(P^T(N - M)\), where \(N - M\) is the number of transit commuters.

To facilitate our analysis, we first define \(M^*\) as follows, corresponding to a given number of parking spaces \(M\):

\[
M^* = \left(\frac{s}{\beta}\right) \cdot P^T(N - M)
\]

A specific \(\tilde{M}^*\) is implicitly determined by the following special instance:

\[
\tilde{M}^* = \left(\frac{s}{\beta}\right) \cdot P^T(N - \tilde{M}^*)
\]

It is evident that \(\tilde{M}^*\) is uniquely defined because \(P^T(\cdot)\) is strictly increasing. As shown in Yang et al. (2013), \(\tilde{M}^*\) represents the threshold number of parking spaces that would push all auto commuters to depart earlier from home such that the last auto commuter just arrives on time (i.e., no late arrival). This special case is shown in Fig. 1(a). Fig. 1(b) presents another case where the parking space constraint is relatively mild, i.e., \(\tilde{M}^* < M < N^A\), such that both late and early arrivals at destination occur. In this case, \(M^*\) represents the number of commuters arriving before or at \(t^*\). Fig. 1(c) illustrates the case where the parking space constraint is so severe, i.e., \(M < \tilde{M}^*\), that all auto commuters are compelled to depart early to secure a parking space, yielding no late arrivals. Note that in Fig. 1, \(s\) is the bottleneck capacity, \(r_1 = \frac{1}{\delta \gamma} s\) and \(r_2 = \frac{1}{\beta} s\) are the departure rates from home for commuters who arrive at the destination before and after \(t^*\) respectively.

2.3. Parking reservation without expiration time

In Yang et al. (2013), a supply of both reserved and unreserved parking spaces is proposed to reduce total travel cost. In their study, the parking spaces will be reserved to commuters no matter when they reach the parking spaces. Following Yang
et al. (2013), we denote the numbers of reserved and unreserved parking spaces to be $M_r$ and $M_u$ respectively ($M_r + M_u = M$), and label the auto commuters with and without reservation as $r$-commuters and $u$-commuters. At equilibrium, assuming that the parking space constraint is binding, the numbers of $r$-commuters and $u$-commuters will be $M_r$ and $M_u$ respectively.

Depending on specific values of $M_r$, $M_u$ and $M$, various departure patterns of commuters emerge at equilibrium. The scenarios (I, II and III) are depicted in Fig. 2 while the parking supply regions yielding the scenarios are depicted in Fig. 3(a). Fig. 2(a) and (b) correspond to Scenario I, where the departure of $u$-commuters and $r$-commuters are completely separated and thus there is no flow interaction between these two classes of commuters. Fig. 2(d) and (f) correspond to Scenarios II and III respectively where the first $r$-commuter arrives at the bottleneck before the queue of $u$-commuters clears. In the former, all $u$-commuters arrive before $t^*$ while in the latter some $u$-commuters arrive late. Fig. 2(c) and (e) correspond to the boundary cases between Scenarios I and II, and between Scenarios II and III respectively. In all scenarios, the times for the first and last $u$-commuters to arrive at their destination are respectively:

$$t_{1e}^0 = t^* - \frac{M_u}{s}, \quad t_{1e}^1 = t^* - \frac{M_r}{s} + \frac{M_u}{s}.$$  \hspace{2cm} (7)

The time for the first $r$-commuter to arrive at the destination is $t_{r1}^0 = t^* - \deltaM'/\beta s$ for Scenario I and is identical to $t_{1e}^0$ for Scenarios II and III. The times for the last $r$-commuter to arrive at the destination in Scenario I and Scenario II or III are

$$t_{r2}^0 = t^* + \frac{\delta M'_r}{\beta s}, \quad t_{r2}^1 = t^* - \frac{M_u}{s} + \frac{M}{s}.$$  \hspace{2cm} (8)

respectively. Note that, in Scenario I $t_{r1}^1 \geq t_{r2}^1$ while in Scenarios II and III $t_{r1}^1 < t_{r2}^1$. As shown in Yang et al. (2013), the optimal combination of reserved and unreserved parking supply should yield an equilibrium flow pattern in Scenario I, i.e., Fig. 2(a)–(c). The total travel cost in Scenarios II or III is higher than that in the boundary case between Scenarios I and II (depicted by Fig. 2(c)).
This paper considers parking reservations with expiration times. In this case, commuters with a reservation have to arrive at the parking spaces before a predetermined expiration time. Otherwise, the parking spaces will no longer be reserved to them. To take advantage of the reservations, \textit{r}-commuters need to arrive at the parking spaces before their expiration times. It is thus very likely that different expiration times will lead \textit{r}-commuters to depart at different time intervals, thereby further reducing congestion at the bottleneck.

In a general parking reservation scheme where reservations expire at different time points, the number of reserved parking spaces associated with the \( i \)-th expiration time \( t_i \) is denoted by \( M_{ri} \), where \( M_{ri} > 0 \), \( i = 1, 2, \ldots, n \), \( t_1 < t_2 < \cdots < t_n \), and \( \sum_{i=1}^{n} M_{ri} = M \). We also denote the arrival times at the destination of the first and last \( r \)-commuters with expiration time \( t_i \) by \( t_{is} \) and \( t_{ie} \).

As mentioned in Section 2.3, under the parking space constraint, the earliest \( u \)-commuter departs from home and arrives at the destination at \( t_{0s} = t' - M/s \), and the last \( u \)-commuter (\( M \)-th \( u \)-commuter) will arrive at the destination at \( t_{0e} = t' - M/s + M'/s \). In the following analysis, we focus on the case that \( t_i > t_{is} \) and thus \( t_{is} < t_{ie} \), otherwise the expirable parking reservations for \( r \)-commuters become meaningless in terms of further reduction of bottleneck congestion. Under this setting, all \( r \)-commuters will choose to travel by auto mode and arrive at the parking spaces before their expiration times because this way they will enjoy a lower travel cost than \( u \)-commuters. The parking reservation with identical or differentiated expiration times will be further discussed in more detail in the following two sections.

Fig. 2. Possible scenarios of auto commuting equilibrium under non-expirable parking reservation (Yang et al., 2013).
3. Parking reservation with identical expiration time (identical scheme)

In this section, we examine the simplest case in which all parking reservations have the same expiration time, i.e., \( n = 1 \) and \( M = M' \). All commuters with parking reservation need to arrive at the parking spaces before the identical expiration time, \( t_1 \). Since the highway bottleneck capacity is limited to \( s \), for a given time interval, the number of parking spaces reserved to commuters should be less than or equal to the maximum number of commuters that can pass the bottleneck. We thus have the following assumption:

**Assumption 1.** The number of reserved parking spaces does not exceed the cumulative capacity of the highway bottleneck between \( t_1 \) and \( t_1 \), i.e.,

\[
M' \leq (t_1 - t_1) s.
\]

(9)

where \( t_1 \) is defined by Eq. (7) and \( t_1 \) is the identical expiration time.

**Assumption 1** ensures that all the commuters with parking reservation can pass the highway bottleneck before the expiration time. This is necessary for optimal design of the parking reservation scheme, and by this assumption we avoid the tedious consideration of undesirable equilibrium flow patterns. From Eq. (9), we have \( t_1 \geq t_1 = t_1 = t_1 \). Indeed, \( t_1 \) is the early bound of the expiration time, \( t_1 \), under which the last \( r \)-commuter will pass the bottleneck just at \( t_1 \). Under the parking reservation scheme with identical expiration time, various flow patterns can appear at equilibrium, depending on the combination of \( M' \), \( M' \), \( M \), and \( t_1 \). To further our analysis, we divide the parking supply domain \( (M, M') \) into four regions, which are summarized in Table 1 and shown in Fig. 3(b). The parking supply domain in Fig. 3(b) indeed can be obtained by splitting region I in Fig. 3(a) into two regions, i.e., regions I and II in Fig. 3(b). Therefore, in regions I and II of Fig. 3(b), \( t_1 \geq t_1 \), while in regions III and IV, \( t_1 < t_1 \). Given the parking supply regions, the equilibrium flow pattern can be subsequently determined based on the expiration time of parking reservation, \( t_1 \) (see Fig. 4).

For parking supply region I, three possible equilibrium flow patterns can emerge. Flow pattern I(a) appears when \( t_1 \leq t_1 \). In this case, the expiration time of a parking reservation is binding (i.e., \( t_1 = t_1 \)) and commuters arrive earlier

### Table 1
Parking supply and corresponding equilibrium.

<table>
<thead>
<tr>
<th>Region</th>
<th>Parking provision</th>
<th>Range of ( t_1 )</th>
<th>Equilibrium flow patterns in Fig. 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( M &lt; M' )</td>
<td>( t_1 \leq t_1 \leq t' )</td>
<td>I(a)</td>
</tr>
<tr>
<td>II</td>
<td>( \bar{M} &lt; M &lt; \bar{N} )</td>
<td>( t_1 = t_1 )</td>
<td>II(a)</td>
</tr>
<tr>
<td>III</td>
<td>( \bar{M} &lt; M &lt; \bar{N} )</td>
<td>( t_1 \leq t_1 )</td>
<td>III(a)</td>
</tr>
<tr>
<td>IV</td>
<td>( M &lt; M' &lt; \bar{N} )</td>
<td>( t_1 \leq t_1 )</td>
<td>IV(a)</td>
</tr>
</tbody>
</table>

Note: \( t_1 = t' + \delta M' / s \) and \( t_1 = t' - M' / s + M / s \).
than $t^*$ either due to the parking competition ($u$-commuters) or the parking reservation expiring before $t^*$ ($r$-commuters).

Flow pattern I(b) arises when $t^* < t_1 < t_{1,1}$. In this case, some $r$-commuters arrive later than $t^*$ since the expiration time is extended, i.e., $t_1 > t^*$. Flow pattern I(c) appears when $t_1 > t_{1,1}^*$, which is identical to the flow pattern under the non-expirable parking reservation scheme. In this case, the expiration time is non-binding (i.e., $t_1 > t_{1,1}^*$), and $r$-commuters will all choose to arrive earlier than the expiration time to reduce the late arrival penalty.

For region II, two possible equilibrium flow patterns emerge. Flow pattern II(a) appears when $t_{1,2} < t_1 < t_{1,1}^*$ and is similar to flow pattern I(b) under which the expiration time of the parking reservations is binding or $t_1 = t_{1,1}^*$. Flow pattern II(b) appears when $t_1 > t_{1,1}^*$, which is identical to the flow pattern under non-expirable parking reservations. In this case, the expiration time is non-binding or $t_1 > t_{1,1}^*$, and the last $r$-commuter will choose to arrive earlier than the expiration time to reduce the late arrival penalty.
For regions III and IV, the expiration time of the parking reservation will not be binding or \( t_1 \geq t_2^1 \). In fact, with a large number of \( u \)-commuters, the analysis can be exactly reduced to the one considered in Yang et al. (2013), i.e., parking reservation without expiration time.

**Proposition 1.** Considering an identical parking reservation scheme, in each of the parking supply regions, an expiration time satisfying the following

\[
\tilde{t}_1 \geq \max\{t_{e,1}^1, t_{e,2}^1\}
\]

minimizes the total travel cost, where \( t_{e,1}^1 \) and \( t_{e,2}^1 \) are given by Eq. (8).

**Proof.** See Appendix A.

Given \( M, M' \) and \( M'' \), if Eq. (10) holds, the equilibrium flow pattern will be identical to that under the parking reservation scheme without expiration time. Therefore, Proposition 1 implies that it is socially preferable to set the parking reservation to be non-expirable (the reservation is valid no matter when the commuter arrives, or the expiration time is sufficiently large) if only a single expiration time is allowed. Intuitively, if the expiration time is sufficiently large, the deadweight loss associated with the parking competition among \( r \)-commuters can be eliminated. With an earlier expiration time, commuters with a reservation may be forced to depart early, incurring higher schedule delay.

### 4. Parking reservation with differentiated expiration times (differentiated scheme)

We now investigate parking reservations with differentiated expiration times. It is expected that a well-designed scheme will make \( r \)-commuters with different expiration times stage their departures such that congestion can be reduced. The \( r \)-commuters can be sorted by their expiration times, and the \( i \)-th group of \( r \)-commuters is associated with the expiration time \( e_i \). Similar to Assumption 1 in Section 3, we have the following assumption:

**Assumption 2.** The number of reserved parking spaces with expiration time \( e_i \) does not exceed the cumulative capacity of the highway bottleneck between \( t_{i-1} \) and \( t_i \), i.e.,

\[
M_i^e \leq (t_i - t_{i-1})s,
\]

where \( i = 1, 2, \ldots, n \) and \( t_0 = t_0^0 \).

From Assumption 2, it is straightforward to see that \( \tilde{t}_i \geq \tilde{t}_{i-1} + M_i^e / s \), thus

\[
\tilde{t}_i \geq t_0^1 + \frac{1}{s} \sum_{j=1}^{i} M_j^e
\]

where \( i = 1, 2, \ldots, n \). Note that Assumption 1 is a special case of Assumption 2 when \( n = 1 \). It is worth mentioning that the optimal design of the parking reservation scheme requires \( M_i^e = (t_i - t_{i-1})s \); otherwise, there will be waste of capacity at the highway bottleneck.

#### 4.1. Differentiated scheme with two expiration times

First we look at a differentiated scheme with two expiration times, i.e., \( n = 2 \) and \( M_1^e + M_2^e = M' \). From Eq. (12), we have \( \tilde{t}_1 \geq t_0^1 + M_1^e / s \) and \( \tilde{t}_2 \geq t_0^1 + M'_r / s = t' - M'' / s + M'/s \). Various equilibrium flow patterns may appear, depending on the combination of \( M, M', M'', t_1 \) and \( t_2 \). Since we are mainly interested in reducing total travel cost, the following discussion will focus on finding the flow pattern incurred by the optimal differentiated scheme with two expiration times under given \( M, M' \) and \( M'' \).

**Lemma 1.** For given \( M, M' \) and \( M'' \), the optimal design of \( M_1^e, M_2^e, \tilde{t}_1 \) and \( \tilde{t}_2 \) must yield \( t_0^1 \geq t_2^1 \).

The inequality, \( t_0^1 \geq t_2^1 \), suggests that the first arrival of the \( r \)-commuter from group 2 is no later than the last arrival of the \( r \)-commuter from group 1. Suppose that \( t_0^1 < t_2^1 \), we can then set a new combination of \( M_1^e, M_2^e, \tilde{t}_1 \) and \( \tilde{t}_2 \) such that \( r \)-commuters in group 1 delay their departure by \( \Delta t = t_2^1 - t_1^e \), and \( r \)-commuters in group 2 maintain their original departure times. This way the total travel cost can be reduced, which contradicts the statement that the original combination of \( M_1^e, M_2^e, \tilde{t}_1 \) and \( \tilde{t}_2 \) is optimal. Lemma 1 indeed implies that under the optimal parking reservation scheme with given \( M \) and \( M'' \), the highway bottleneck is fully utilized between \( t_0^1 \) and \( t_2^1 \).

**Lemma 2.** For given \( M, M' \) and \( M'' \), the optimal design of \( M_1^e, M_2^e, \tilde{t}_1 \) and \( \tilde{t}_2 \) must yield \( t_2^1 > t' \) and \( q(t_2^1) = 0 \).
This lemma can be proved by contradiction. If $t_2 < t^*$ or $q(t_2) > 0$, the second expiration time of parking reservations must be binding, i.e., $t_2 = t^*$. In this case, we can prolong the expiration time such that $r$-commuters in group 2 can reduce travel cost by departing later. This result and the proof are similar to those of Proposition 1. (As stated in Proposition 1, the optimal expiration time should be set to be sufficiently large, thus at equilibrium, the last commuter will arrive later than $t^*$ and face no queue).

Considering parking supply region I, there are only four possible flow patterns (depicted in Fig. 5) satisfying both Lemmas 1 and 2. Note that in Fig. 5, we assume that the expiration times of these two-step parking reservations are well designed thus $t_1 = t_1^* = t^*_2$ and $t_2 = t_2^*$.

**Proposition 2.** For parking supply region I with given $M$, $M'$ and $M''$, the optimal design of $M_1'$, $M_2'$, $t_1$ and $t_2$ must yield the flow pattern depicted in Fig. 5(a).

**Proof.** See Appendix B. □

For parking supply region II, the result and analysis are similar. Given $M$, $M'$ and $M''$, the optimal design of $M_1'$, $M_2'$, $t_1$ and $t_2$ yields a flow pattern similar to that in Fig. 5(a). The difference is that for region II, $M > M''$ holds (in Fig. 5, $M \leq M''$). These results imply that for regions I and II, if we set two expiration times, it is socially preferable to separate the two $r$-commuter groups’ arrivals at the bottleneck completely. In this case, the first $r$-commuter in the second group will arrive at the bottleneck when the last $r$-commuter in the first group just leaves it.

For regions III and IV, under a given number of unreserved parking spaces, $M''$, dividing the reserved parking spaces into more groups with different expiration times will not change the equilibrium flow pattern. Therefore, in this case, the total travel cost cannot be further reduced by differentiating expiration times.

### 4.2. Differentiated scheme with multiple expiration times ($n > 2$)

For given $M$, $M'$ and $M''$ (in parking supply regions I and II), by repeating the above analysis of the scheme with two expiration times, we can obtain the flow pattern under the optimal design of $M_i'$ and $t_i$ for $i = 1, 2, \ldots, n$ for $n > 2$, which is depicted in Fig. 6(a) and (b). As we can observe, $u$-commuters depart earlier than $r$-commuters to compete for parking spaces while $r$-commuters with different expiration times will depart at their corresponding time intervals.

When $n \to \infty$, the flow patterns will become the ones depicted in Fig. 6(c) and (d). We suppose that the expiration times of parking reservations are well designed such that $t_i = t_i^* = t_{i+1}^*$ for $i = 1, 2, \ldots, n - 1$. Without loss of generality, it is assumed that the reserved parking spaces are divided into $n$ groups equally, i.e., all time intervals, $[t_i, t_{i+1}]$ or $[t_i^*, t_{i+1}^*]$ where $i = 1, 2, \ldots, n - 1$, have equal length, and $M_i' = M'/n$ for every $i$. As can be seen in Fig. 6, when $n \to \infty$, the $r$-commuters can be characterized into two classes: those facing queue or no queue at the bottleneck. In Fig. 6(c) and (d), the first $k$ groups of $r$-commuters are those facing no queue while the other $n - k$ groups of $r$-commuters are those facing positive queue. In

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**Fig. 5.** Equilibrium flow patterns satisfying Lemmas 1 and 2 for parking supply region I.
M is the departure rate of these r-commuters, thus they can arrive earlier and incur a lower late arrival penalty. For r-commuters, will face queue at the bottleneck. Further, some of early r-commuters in the last group will face the longest queue, which is given by

\[ t_n = \frac{1}{C_0} M + \frac{1}{C_22} r_1 s, \tag{13} \]

for every \( i \), where \( r_1 \) is the departure rate of these r-commuters. Since \( M'_r = M' / n \), we can readily observe that when \( n \to \infty \), \( q_i \to 0 \). This implies that for the first \( k \) groups, further division of the parking reservation groups will mitigate the queue further at the bottleneck.

Now we turn to the r-commuters facing queue at the bottleneck. We begin with the r-commuters in the last group. All of r-commuters in the last group will arrive at the destination between \( t_{n-1} \) and \( t_n \) and some of these commuters will leave home earlier than \( t_{n-1} \), thus they can arrive earlier and incur a lower late arrival penalty. For r-commuters in the second last group, they will arrive at the destination between \( t_{n-2} \) and \( t_{n-1} \). The last r-commuter in this group has to depart earlier than the first r-commuter in the last group to ensure that he or she can arrive at the parking space before its expiration time, \( t_{n-1} \). Repeating this analysis group by group in a reverse order of arrival time, we will see that the r-commuters in the earlier group are driven to depart earlier by the r-commuters in the latter group, thereby forming a queue at the bottleneck. This implies, for these \( n - k \) groups of r-commuters, further division of the parking reservation groups would not help to reduce the queue.

Indeed, all the late r-commuters, denoted by \( M'_r \), will face queue at the bottleneck. Further, some of early r-commuters will be influenced by those late r-commuters and have to face queue at the bottleneck. The number of the influenced early r-commuters, denoted by \( M'_e \), is given by \( M'_e = M'_r / \beta \), which evidently increases with \( M'_r \). Under the optimal expirable parking reservation scheme, \( M'_e + M'_r = M_n^* \) should hold, where \( M_n^* \) is determined by Eq. (19) in Section 4.3.

4.3. Optimal design of differentiated scheme (\( n \geq 2 \))

As discussed in Section 4.2, the further division of groups (increasing \( n \)) will not influence the last \( n - k \) groups. Now since we focus on the optimal design of the differentiated scheme, without loss of generality, we designate these \( n - k \) groups as only one group, i.e., the last group. Further, the rest reserved parking spaces are classified into \( n - 1 \) groups equally. Then, we will have \( t_n = t' + \frac{1}{C_22} M_n^* \) and \( t_{n-1} = t' + \frac{1}{C_22} M_{n-1}^* \), and \( M'_r = (M' - M_n^*) / (n - 1) \) where \( i = 1, 2, \ldots, n - 1 \). Note that from now on, the number of reserved parking spaces for the last group might be different from the other groups. Given \( M \) and \( n \), the optimal design of the scheme will minimize the total travel cost under the flow pattern depicted in Fig. 6(a) and (b), which is:

Fig. 6. Flow patterns under differentiated scheme with multiple expiration times.
\[
TC_n = M^u \times P^u + M^d_n \times \left( \frac{M^d}{s} \right) + (M^d - M^u) \times \left( \frac{M^d}{s} + \frac{1}{2} \frac{n \beta - M^u}{s} \right). 
\]

where \(M^u + M^d = M\) and \(0 \leq M^d_n < M^d \leq M\). In Eq. (14), the three terms correspond to the total travel costs of the \(u\)-commuters, the last group of \(r\)-commuters, and the rest \(r\)-commuters, respectively.

We begin with defining two critical numbers of parking spaces, \(M_1\) and \(M_2\), which solves:

\[
P^\ast (N - M_1) = (2 - \frac{1}{2} \frac{\beta}{\beta}) \frac{M_1}{s}; \quad P^\ast (N - M_2) = (2 - \frac{\beta}{\beta}) \frac{M_2}{s},
\]

and for given \(M_1 < M < M_2\), we further define the following critical number:

\[
n_m = \left[ 1 - \beta \left( 2 - \frac{n P^\ast (N - M)}{\beta M} \right) \right]^{-1}.
\]

Note that when \(M_1 < M < M_2\), we have \(n_m > 2\). Based on the above two definitions, we are now ready to present the following proposition.

**Proposition 3.** Given that the total parking supply is insufficient, i.e., \(M < N^A\), under the optimal differentiated scheme with \(n \geq 2\),

(i) If \(M \leq M_1\), all parking spaces should be reserved to commuters.

(ii) If \(M_1 < M < M_2\), all parking spaces should be reserved to commuters when \(n > n_m\); some parking spaces should be retained for open competition when \(n \leq n_m\).

(iii) when \(M \geq M_2\), some parking spaces should be retained for open competition.

**Proof.** See Appendix C. □

Note that in Proposition 3, \(n\) should be an integer. The proposition implies that, (i) when the parking space constraint is relatively severe, i.e. \(M \leq M_1\), competition for parking is fierce and \(u\)-commuters have to depart very early to secure a parking space and thus encounter very large schedule delays. Therefore, all parking spaces should be reserved to eliminate the high schedule cost from parking competition. (ii) When the parking space constraint is relatively mild, i.e. \(M_1 < M < M_2\), competition for parking is less severe and the schedule delay due to early arrival encountered by \(u\)-commuters is not substantial. Therefore, if the number of expiration times cannot be large enough, i.e., \(n \leq n_m\) (this might be more realistic and practical in reality), retaining some parking spaces unreserved would separate the departures of commuters to reduce the queuing delay. The reduction in queuing delay outweighs the increase in schedule delay cost. However, if the number of expiration times can be designed to be sufficiently large, i.e., \(n > n_m\), it is always socially preferable to make all parking spaces reserved. This is because the division of parking spaces into \(n\) groups reduces the queuing delay substantially when \(n > n_m\). The marginal reduction in queuing delay cost by converting one reserved parking space to an unreserved one is limited and indeed outweighed by the increase in the marginal schedule delay cost. (iii) When the constraint is even milder, i.e., \(M \geq M_2\), it is always preferable to retain some spaces unreserved.

To optimally design the parking reservation scheme with given \(n\) expiration times, we define the following two critical values of the parking supply:

\[
P^\ast (N - M_{n,1}) = (2 - \frac{\delta n - 1}{\beta} \frac{M_{n,1}}{s}),
\]

and

\[
P^\ast (N - M_{n,2}) = \left( 1 - \frac{\delta}{\gamma} \frac{1}{2} \frac{n \beta}{\beta} \frac{M_{n,2}}{s} \right) \left( \frac{M_{n,2}}{s} \right),
\]

From Eqs. (17) and (18), and the fact that \(N^A = \frac{\mu^u (N - M^u)}{\mu^u + M^u}\), it can be verified that \(M_{n,1} < M_{n,2} < N^A\). Given \(n\), based on the critical total parking supplies defined above, we can divide the parking supply into three intervals: \((0, M_{n,1})\), \((M_{n,1}, M_{n,2})\) and \((M_{n,2}, N^A)\). Given \(M\) and \(n\), with Eq. (14), we can derive the unique optimal combination of \(M^u\) and \(M^d_n\) as follows:

\[
M^u, M^d_n = \begin{cases} 
M \left( 1 - \frac{n - 1}{\beta} \frac{M}{\beta} \right) & 0 < M \leq M_{n,1} \\
\left( 2 - \frac{n - 1}{\beta} \frac{M}{\beta} \right) \frac{M^u}{\gamma} \left( 1 - \frac{n - 1}{\beta} \frac{M^u}{\beta} \right) M_{n,1} < M \leq M_{n,2} \\
M - \frac{1}{\beta} \frac{M^u}{\gamma} - M \left( M - \frac{M^u}{\gamma} \right) & M_{n,2} < M < N^A 
\end{cases}
\]
where $P^f = P^f(N - M)$. It is worth mentioning that the derivation of Eq. (19) relies on the fact that $n > 2$, however, this result holds for $n = 1$ where $M = M'$. Also note that $M^{*} = M - M'^{*}$ and for $n \geq 2$ we have $M^{*} = (M^{*} - M'^{*})/(n - 1)$ for $i = 1, 2, \ldots, n - 1$.

From Eq. (19), we can see that when $M \in (0, M_{n,1}]$, it is socially preferable to make all parking spaces reserved. According to Proposition 3, we further know that for the parking supply $M \in (0, M_{n,1}]$, either $M \leq M_1$ or $M_1 < M < M_2$ and $n > n_0$ occurs. From Eq. (17), it can be verified that $M_{n,1}$ increases with respect to $n$. We thus have the following proposition.

**Proposition 4.** Under the optimal expirable parking reservation scheme with $n$ expiration times, all parking spaces should be reserved when $M \in (0, M_{n,1}]$. This interval increases as the number of expiration times increases.

This proposition implies that although for some parking supply interval, it might be socially preferable to retain some parking spaces for open competition (unreserved), this interval, i.e., $(M_{n,1}, M_{n,2}]$, diminishes as the capability or flexibility of differentiating expiration time increases. In addition, when $n \to \infty$, for the parking supply interval where $M > M_{n,2}$, we have $M^{*} \to M$. (This can be verified according to Eq. (19) when $n \to \infty$).

### 5. Efficiency of the parking reservation scheme

In this section, we examine the efficiency of the expirable parking reservation scheme and compare it with the non-expirable one in [Yang et al. (2013)](#). Let $\theta_n$ denote the maximum percentage reduction of total travel cost of auto commuters, which is defined as follows:

$$
\theta_n := \frac{P^f \cdot M - \min \{TC_n\}}{P^f \cdot M},
$$

where $P^f = P^f(N - M)$ (constant for given $M$). In Eq. (20), $P^f \cdot M$ represents the total cost of auto commuters under no policy intervention and all parking spaces being unreserved; $\min \{TC_n\}$ is the minimum total cost achieved by implementing the optimal parking reservation scheme with $n$ different expiration times. From Proposition 1, we know that the efficiency of the optimal non-expirable parking reservation scheme in [Yang et al. (2013)](#) is the same as the that of the optimal identical scheme where $n = 1$, thus in the following analysis, $\theta_1$ indeed represents both the efficiencies of the non-expirable scheme and identical scheme.

Based on the results in Section 4, the percentage reduction, $\theta_n$, can be determined as follows:

$$
\theta_n = \begin{cases} 
1 - \frac{1}{2} \left(2 - \frac{n+1}{n} \frac{PD}{P^f} \right) & 0 < M \leq M_{n,1} \\
\frac{1}{2} \left(2 - \frac{n+1}{n} \frac{PD}{P^f} \right) & M_{n,1} < M \leq M_{n,2} \\
\frac{1}{2} \left(1 - \frac{n+1}{n} \left(\frac{PD}{P^f} - 1\right) \right) \left(1 - \frac{PD}{P^f} \right) & M_{n,2} < M < N^A
\end{cases}
$$

where $P^f = P^f(N - M)$. Here, for simplicity, we treat $n$ as a continuous variable, and differentiating Eq. (21) with respect to $n$ yields:

$$
\frac{\partial \theta_n}{\partial n} = \begin{cases} 
\frac{1}{2} \left(2 - \frac{n+1}{n} \frac{PD}{P^f} \right) & 0 < M \leq M_{n,1} \\
\frac{1}{2} \left(2 - \frac{n+1}{n} \frac{PD}{P^f} \right)^2 \frac{1}{n} > 0 & M_{n,1} < M \leq M_{n,2} \\
\frac{1}{2} \left(1 - \frac{n+1}{n} \left(\frac{PD}{P^f} - 1\right) \right) \left(1 - \frac{PD}{P^f} \right) & M_{n,2} < M < N^A
\end{cases}
$$

Further, we can derive the second-order derivative of Eq. (21), and it can be shown that $\partial^2 \theta_n/\partial n^2 < 0$. Note that this result still holds where $n$ is discrete.

**Proposition 5.** The percentage of cost reduction, $\theta_n$, increases with the number of expiration times, i.e., $n$. Moreover, the rate of increase would decrease as $n$ becomes larger.

**Proposition 5** indeed implies more travel cost can be reduced and efficiency of the reservation scheme can be improved if we are able to set more expiration times and groups. However, as $n$ becomes larger, the marginal benefit would diminish. It is certainly not practical to set $n \to \infty$. Fortunately, it is likely that a reasonably large $n$ would yield similar efficiency as that with $n \to \infty$, as shown below.

We define the efficiency loss for the optimal parking reservation scheme with $n$ expiration times as follows:

$$
l_n := \frac{\theta_{\infty} - \theta_n}{\theta_{\infty}},
$$

where $\theta_{\infty}$ is the percentage of cost reduction when $n \to \infty$, and $\theta_n - \theta_{\infty}$ indeed represents the percentage of cost reduction that the scheme with $n$ steps loses compared with the scheme with infinity steps. Detailed expression of the efficiency loss, $l_n$, and the proof of the following proposition are given in Appendix D.
Table 2
The efficiency and its lower and upper bounds in each interval of parking supply.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$\delta_n$</th>
<th>$\delta^d_n$</th>
<th>$\delta^d_u_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M \leq M_{n,1}$</td>
<td>$1 - \frac{1}{2} \left( \frac{2 - \frac{\gamma}{\gamma + 1}}{ \gamma + 1 } \right)$</td>
<td>$\frac{1}{2}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$M_{n,1} &lt; M \leq M_{n,2}$</td>
<td>$\frac{1}{2} \left( \frac{2 - \frac{\gamma}{\gamma + 1}}{ \gamma + 1 } \right)^{-1} - \frac{1}{2} \left( \frac{2 - \frac{\gamma}{\gamma + 1}}{ \gamma + 1 } \right)$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$M_{n,2} &lt; M \leq N^a$</td>
<td>$\frac{1}{2} \left( \frac{2 - \frac{\gamma}{\gamma + 1}}{ \gamma + 1 } \right) \left( \frac{2 - \frac{\gamma}{\gamma + 1}}{ \gamma + 1 } \right) - \frac{1}{2}$</td>
<td>$0$</td>
<td>$\left( \frac{2 - \frac{\gamma}{\gamma + 1}}{ \gamma + 1 } \right)^{-1}$</td>
</tr>
</tbody>
</table>

Note: $P^d = P^d(N - M)$. 

Proposition 6. The efficiency loss $b_n < \frac{1}{\gamma}$.

From the above proposition, we know that, for example, $n = 5$, will yield an efficiency loss less than 20% when $n = 10$ will yield an efficiency loss less than 10%. This implies that a reasonably large $n$ would reduce travel cost satisfactorily and efficiency is close to that under the reservation scheme with $n \to \infty$.

Similar to Yang et al. (2013), for each interval of the parking supply, the lower and upper bounds, $\theta^u_n$ and $\theta^d_n$ of $\theta_n$ can also be found. Under the assumption $\gamma > \beta > 0$, we can further bound $\theta^d_n$ and $\theta^u_n$. These results are all summarized in Table 2. The case considered in Yang et al. (2013) is a special case of our analysis where $n = 1$.

Table 2 and Eq. (19) together tell us that, for a scheme with $n$ expiration times, when it is socially preferable to reserve all parking spaces to commuters ($M \leq M_{n,1}$), the corresponding percentage of cost reduction is at least 50%; for all other parking supply intervals, it is socially preferable to retain some parking spaces unreserved, the corresponding percentage of cost reduction is always less than 50%. To summarize, we state the following proposition.

Proposition 7. When $M^* = M$, $M^d = 0$, the percentage reduction of the total cost is at least 50%; When $M^* < M$, $M^d > 0$, the percentage of total cost reduction is at most 50%.

This result extends that in Yang et al. (2013) to a more general case where expirable parking reservation is considered. Note that, as Proposition 4 states, as the number of expiration times increases, the parking supply interval where $M^* = M$, $M^d = 0$ increases, thus the parking supply interval where percentage of cost reduction is higher than 50% increases.

6. Conclusion and discussions

6.1. Conclusion

This study proposes and examines an expirable parking reservation scheme to manage the morning commute when the parking supply at the destination is insufficient. Under the scheme, commuters with a parking reservation must arrive at the parking spaces before a predetermined expiration time; otherwise the parking spaces are no longer reserved for them. If the expiration time is identical, we found that it is socially preferable not to allow parking reservations to be expirable. However, if multiple differentiated expiration times are allowed, the expirable reservation scheme can be designed to further reduce the total travel cost as compared to the non-expirable reservation scheme in Yang et al. (2013).

When parking space constraint is severe, all parking spaces should be reserved to commuters. When parking space constraint is relatively mild, it may be socially preferable to retain some parking spaces for open competition. The number of unreserved parking spaces depends on the number of differentiated times allowed in the reservation scheme. Further, the parking supply interval where it is socially preferable to make all parking spaces reserved increases as the number of expiration time increases.

Generally speaking, differentiated reservation schemes have higher efficiency than the scheme with identical expiration time, and a higher level of expiration time differentiation, i.e., larger $n$, will lead to a higher reduction in travel cost. However, the marginal benefit of expiration time differentiation diminishes. Compared with the scheme with $n \to \infty$, which yields the highest efficiency, the loss of efficiency is $b_n < \frac{1}{\gamma}$ This suggests that a reasonably large number of expiration times (e.g., 5) can lead to satisfactory results (with less than 20% efficiency loss).

In Yang et al. (2013), it is found that when the system optimum requires all parking spaces to be reserved, the maximum percentage of total cost reduction is always more than or equal to 50%. In all other cases, the maximum percentage of total cost reduction is always less than 50%. This still applies when we consider expirable parking reservations with differentiated expiration times, independent of the level of expiration time differentiation.

6.2. Some further discussions

This study is an attempt to explore the potentials of the differentiated parking reservations in managing the morning commute in a dynamic context. To implement such strategies, many practical issues have to be addressed. They include, but not limited to: How to realize parking reservation? (e.g., online auction, booking similar to industries such as airline, railway, hotel and entertainment) Should the parking reservation be long-term or short-term? When commuters are heterogeneous in their values of travel time and schedule delay, who should obtain parking reservation with more flexibility (larger
expiration time)? An accompanying paper, Liu et al. (in preparation), develops an expirable parking permit scheme to realize or implement the expirable parking reservations discussed here when user heterogeneity exists. They show that such a permit scheme can realize the optimal allocation of expirable parking reservations among commuters, and the optimal design of the scheme does not require commuters’ value of time information, and the performance of the scheme is robust to the variation of commuters’ value of time.

While the current paper focuses on managing parking competition and roadway congestion, further research may consider incorporating the parking cruising time or parking search friction. Generally, parking cruising time is tightly related to parking occupancy. In the context of this paper, for an individual commuter, the parking cruising time can be linked to the number of the remaining parking spaces at the commuter’s arrival time, given the total number of available parking spaces. Therefore, the parking cruising time indeed can be incorporated into our current framework by introducing a parking cruising time function, which decreases with the number of remaining parking spaces. The parking cruising time, as part of travel time, would affect commuters’ departure time choices. For instance, an earlier commuter would probably experience a lower cruising time.

If parking cruising is taken into account, the proposed differentiated parking reservations also have the potential to reduce deadweight loss due to cruising. With the new positioning and communication technologies, it is possible for the central authority to assign a specific parking space to a commuter after he or she makes a reservation. At the same time, the location information of the reserved parking space will be sent to the commuter. Then, commuters need no longer to cruise for their reserved parking spaces.

Further research may also introduce differentiated parking reservation fees (with respect to reservations with differentiated expiration times) to avoid commuters’ wasteful competition for better reservations (i.e., reservations with larger expiration times), and reduce the risk of no show. In the current paper, all parking reservations are free of charge. Commuters would compete for reservation with larger expiration time, since it may reduce their travel cost. This competition may not only increase individual commuter’s inconvenience, but also lead to undesired system issue. For example, if all reservations are made through Internet, a lot of commuters may log into the online reservation system at the same time after it opens, and this may lead to system breakdown. Besides, if the reservations are free, no show is more likely to occur, which might lead the system to lose efficiency. Lastly, it worth mentioning that the properly designed parking reservation fees would reflect the travel cost saving from the reservation compared to competing for parking or taking transit.

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Appendix A. Proof of Proposition 1

For parking supply regions I and II, when \( t_1 \leq t_{e1}^1 = t^* + \frac{M^u}{\beta M^*} \), the total travel cost is

\[
TC_1^{II}(t_1) = P T + M^u + \beta (t^* - t_{e1}^1) \cdot M^*,
\]

(24)

where \( P T \) is the travel cost of u-commuters (equal to the transit cost), and \( t_{e1}^1 = t_1 - M^*/s \) is the arrival time of the first r-commuter. From Eq. (24), we can further obtain:

\[
\frac{dT C_1^{II}(t_1)}{dt_1} = -\beta < 0.
\]

(25)

When \( t_1 \leq t_{e1}^1 = t^* + \frac{M^u}{\beta M^*} \), the total travel cost is always equal to \( TC_1^{II}(t_1) = t^* + \frac{M^u}{\beta M^*} \).

For parking supply regions III and IV, the total travel cost is given as follows:

\[
TC_1^{III/IV}(t_1) = P T + M^u + \frac{M - M^u}{s} \cdot M^*,
\]

(26)

where \( P T \) is the travel cost of u-commuters (equal to the transit cost) and \( \gamma(M - M^u)/s \) is travel cost of r-commuters. In these regions, total travel cost is constant and independent of \( t_1 \).

As mentioned in Section 3, in regions I and II, \( t_{e1}^2 \geq t_{e2}^2 \) while in regions III and IV, \( t_{e1}^2 < t_{e2}^2 \). In the former, from (25) and the fact that the total travel cost is equal to \( TC_1^{II}(t_1) = t_{e1}^1 \) when \( t_1 > t_{e1}^1 \), it can be readily observed that the minimum total travel cost is achieved when \( t_1 \geq t_{e1}^1 = \max(t_{e1}^1, t_{e2}^1) \). In regions III and IV, the total travel cost is constant. Also note that \( t_{e2}^2 \) is the early bound of the expiration time, \( t_{e1}^2 \), in regions III and IV, we then have \( t_1 \geq t_{e2}^2 = \max(t_{e1}^1, t_{e2}^2) \). Therefore, Proposition 1 is proved.
Appendix B. Proof of Proposition 2

Proof. For the equilibrium flow patterns depicted in Fig. 5(b)–(c), total travel cost can be further reduced by converting some reserved parking spaces with expiration time $t_1$ into those with $t_2$. Doing so will transfer some r-commuters from group 1 to group 2 and shift their departures to later times as depicted by the blue dotted lines in Fig. 5(b)–(c). Clearly, the queuing delay of these commuters will be reduced while the costs for all other commuters remain the same as before. Thus, the total travel cost can be further reduced.

Now we prove that the flow pattern depicted in Fig. 5(d) is not optimal either. For given $M$ and $M^0$, the total travel cost under the flow pattern depicted in Fig. 5(a) can be written as a function of $M_1'$ as follows:

$$T_{C_2}(M_1') = M^0 \cdot P^T + M_1' \left( \frac{M^0 - M_1'}{s} + \frac{\beta M_1'}{s} \right) + (M^0 - M_1') \left( \frac{M^0 - M_1'}{s} \right).$$

(27)

where $M_1' > 0$. The first, second, and third terms in (27) are the travel costs of u-commuters, r-commuters in group 1 and group 2 respectively. The flow pattern depicted in Fig. 5(d) is identical to that under the limiting parking reservation scheme where $M_1' = 0$ and $M_2' = M'$. Therefore, the total travel cost in this case is equal to $T_{C_2}(0)$. Furthermore, we have:

$$\frac{dT_{C_2}(M_1')}{dM_1'} = -\delta \frac{M^0}{s} + 2\beta \frac{M_1'}{s}, \quad \text{and} \quad \frac{d^2T_{C_2}(M_1')}{d(M_1')^2} = 2\beta \frac{1}{s} > 0.$$

(28)

$$\frac{dT_{C_2}(M_1')}{dM_1'} |_{M_1'=0} = -\delta \frac{M^0}{s} < 0.$$

(29)

Therefore, a small increase in $M_1'$ at $M_1' = 0$ will lead to a decrease in total travel cost. Since total travel cost under the equilibrium flow pattern depicted in Fig. 5(d) is equal to $T_{C_2}(0)$, this flow pattern is also not the one under the optimal design of $M_1', M_2', t_1^*$ and $t_2^*$.

Appendix C. Proof of Proposition 3

Proof. Rewriting Eq. (14) as a function of $M'$ and $M_n'$, we have:

$$T_{C_n} = (M - M') \times P^T + M_n' \times \left( \frac{M_n'}{s} \right) + (M' - M_n') \times \left( \frac{M_n' + 1}{n-1} \frac{n}{\beta} \frac{M_n' - M_n'}{s} \right).$$

(30)

where $P^T = P^T(N-M)$ and $0 < M_n' < M' \leq M$. Under the optimal design of scheme, the earliest departure will be no earlier than $t_1^* = \frac{M_1'}{\beta}$, which is defined by Eq. (7). Therefore,

$$t^* - \frac{\delta M_1'}{\beta} - \frac{M^0 - M_1'}{s} - \frac{M - M^0}{s} \geq t^* - \frac{M^0}{s},$$

or equivalently, $M_n' \geq \frac{1}{2} (M - M^0)$. The optimal design of the parking reservation scheme will minimize Eq. (30) under the constraints of $M' \leq M$ and $M_n' \geq \frac{1}{2} (M - M^0)$. Differentiating Eq. (30) with respect to $M'$ and $M_n'$ respectively yields:

$$\frac{dT_{C_n}}{dM'} = -P^T + \frac{\delta M_n'}{s} + \frac{n}{n-1} \frac{\beta M_n' - M_n'}{s} ; \quad \frac{dT_{C_n}}{dM_n'} = \frac{\delta M_n'}{s} - \frac{n}{n-1} \frac{M_n' - M_n'}{s}.$$

(31)

Also we know from Eqs. (17) and (18) that

$$M \leq M_{n1} \iff P^T \geq \left( 2 - \frac{\delta n - 1}{\beta n} \right) \frac{M_n'}{s} ; \quad M \leq M_{n2} \iff P^T \geq \left( 1 + \frac{\delta}{2} - \frac{\delta n - 1}{\beta n} \right)^{-1} \frac{M_n'}{s},$$

(32)

and vice versa. From Eqs. (31) and (32) and the fact that $M' \leq M$ and $M_n' \geq \frac{1}{2} (M - M^0)$, we obtain the following condition under the optimal design of parking reservation scheme:

$$\left\{ \begin{array}{l} \frac{dT_{C_n}}{dM} < 0, \quad 0 < M \leq M_{n1} \\ \frac{dT_{C_n}}{dM} = 0, \quad M_{n1} < M \leq M_{n2} \\ \frac{dT_{C_n}}{dM} > 0, \quad M_{n2} < M < N^A \end{array} \right.$$

As a result, the optimal combination of $M'$ and $M_n'$ can be determined, which are given in the main text by Eq. (19). With Eq. (15), we have:

$$M \leq M_1 \iff P^T \geq \left( 2 - \frac{1}{\beta} \right) \frac{M_n'}{s} ; \quad \text{and} \quad M > M_1 \iff P^T < \left( 2 - \frac{1}{\beta} \right) \frac{M_n'}{s}.$$

(33)
Now we are ready to prove the two cases in Proposition 3: (i) \( M \leq M_1 \), (ii) \( M_1 < M < M_2 \), and (iii) \( M \geq M_2 \).

**Proof of (i).** Since \( n \geq 2 \), according to Eqs. (32) and (33) we know that \( M \leq M_1 \leq M_{n,1} \), from Eq. (19), we have \( M^{**} = M \).

**Proof of (ii).** We still consider \( n \geq 2 \). For the case with \( M_1 < M < M_2 \), it can be verified by looking at Eq. (16) that \( n_M > 2 \). When \( M > M_1 \) and \( n > n_M \), \( M \leq M_{n,1} \), holds, from Eq. (19), \( M^{**} = M \) holds. When \( M > M_1 \) and \( n \leq n_M, M > M_{n,1} \), holds, from Eq. (19), we know \( M^{**} < M \).

**Proof of (iii).** When \( M \geq M_2, M > M_{n,1} \), holds, from Eq. (19), we have \( M^{**} < M \). \qed

**Appendix D. Proof of Proposition 6**

**Proof.** From Eqs. (21) and (23), efficiency loss can be determined as follows:

\[
I_n = \begin{cases} 
\frac{1}{2} \frac{\delta M}{\delta T} \left( \frac{1 - \frac{\delta M}{\delta T}}{\frac{1}{2} \frac{\delta M}{\delta T}} \right)^{-1} & 0 < M \leq M_{n,1} \\
\frac{1}{2} \left( \frac{\delta M}{\delta T} - 1 \right) \left( 1 - \frac{\delta M}{\delta T} \right)^{-1} & M_{n,1} < M \leq M_{n,2} \\
\left( \frac{\delta M}{\delta T} - 1 \right) \left( 1 - \frac{\delta M}{\delta T} \right)^{-1} & M_{n,2} < M < N^A 
\end{cases}
\]

where \( P^T = P(N - M) \). For each interval of parking supply, the efficiency loss in Eq. (34) can be regarded as a coefficient times \( \frac{T}{n} \). It suffices to show all the three coefficients are less than one. Firstly note that the parking space constraint implies that \( \frac{T}{n} > \frac{\delta M}{\delta T} \). Therefore,

\[
\frac{sP^T}{\delta M} > 1 \quad \text{and} \quad 1 - \frac{\delta M}{sP^T} > 0.
\]

For \( 0 < M \leq M_{n,1} \), we then have:

\[
1 - \frac{\delta M}{sP^T} < 1 - \frac{\delta M}{\frac{sP^T}{2}} \Rightarrow 1 - \frac{\delta M}{2 \frac{\delta M}{sP^T}} < \frac{1}{2} \frac{\delta M}{\delta T} \left( \frac{1 - \frac{\delta M}{\delta T}}{\frac{1}{2} \frac{\delta M}{\delta T}} \right)^{-1} < 1.
\]

For \( M_{n,1} < M \leq M_{n,2} \), we note that \( \frac{\delta}{\delta T} \), and thus

\[
\frac{\delta}{\delta T} < 1 \Rightarrow 1 - \frac{\delta}{\delta T} < 2 - \frac{\delta}{\delta T} < 2 \frac{\delta}{\delta T} + \frac{\delta}{\delta T} \Rightarrow 2 \frac{\delta}{\delta T} + \frac{\delta}{\delta T} < 1.
\]

For \( M_{n,2} < M < N^A \), from Eq. (18), we know that

\[
\frac{sP^T}{\delta M} < \left( 1 - \frac{\delta}{\delta T} \right) \left( \frac{1}{2} \frac{\delta M}{\frac{\delta M}{sP^T}} \right)^{-1} \Rightarrow \frac{sP^T}{\delta M} < \left( 1 - \frac{\delta}{\delta T} \right) < \left( 1 - \frac{\delta}{\delta T} \right)^{-1}.
\]

We then have

\[
\frac{1}{2} \frac{\gamma}{\delta T} \left( \frac{sP^T}{\delta M} - 1 \right) < 1 - \frac{\gamma}{\delta T} \left( 1 - \frac{\delta}{\delta T} \right) \left( 1 - \frac{\delta}{\delta T} \right)^{-1} = 1 - \frac{\gamma}{\delta T} \frac{\delta}{\gamma} < 1
\]

as a result,

\[
\frac{1}{2} \frac{\gamma}{\delta T} \left( \frac{sP^T}{\delta M} - 1 \right) < 1 - \frac{\gamma}{\delta T} \left( \frac{sP^T}{\delta M} - 1 \right) \Rightarrow 1 - \frac{\gamma}{\delta T} \left( \frac{sP^T}{\delta M} - 1 \right) < 1.
\]

Therefore, \( I_n \) in Eq. (34) will be strictly less than \( \frac{T}{n} \). \qed

**References**