Since $P = P_e(x_1) - \rho u_2^2$

$$\frac{1}{\rho} \frac{\partial P}{\partial x_1} = \frac{1}{\rho} \frac{\partial P_e}{\partial x_1} - \frac{\partial u_2^2}{\partial x_1}$$

In external flow (outside B.L.) Bernoulli’s equation applies (inviscid, etc.).

$$P_e + \frac{1}{2} \rho U_e^2 = P_r = \text{constant}$$

$$\frac{\partial P_e}{\partial x_1} + \frac{1}{2} \rho 2U_e \frac{\partial U_e}{\partial x_1} = 0$$

$$\frac{1}{\rho} \frac{\partial P_e}{\partial x_1} = U_e \frac{\partial U_e}{\partial x_1}$$

Combine with x-momentum equation.

Term from $x_i$ - dir integration

$$U_1 \frac{\partial U_1}{\partial x_1} + U_2 \frac{\partial U_1}{\partial x_2} = U_e \frac{\partial U_e}{\partial x_1} + \nu \frac{\partial^2 U_1}{\partial x_1^2} - \frac{\partial u_2^2}{\partial x_1} - \frac{\partial}{\partial x_1} \left( u_1^2 - u_2^2 \right)$$

May be important for flows approaching separation.

We will neglect for well-behaved B-L's.

With $U = \overline{u_1}$, $V = \overline{u_2}$, $\overline{u_1'}v = \overline{u_1u_2}$, $U = U_e = U_\infty$

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = U \frac{\partial U}{\partial x} + \nu \frac{\partial^2 U}{\partial y^2} + \frac{\partial u_1' v}{\partial y}$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

2 equations, 3 unknowns.

U assumed known.

Evaluation of U.
a. External Flows
    - Let $U$ be potential flow velocity on surface for 1$^{\text{st}}$ approximation.

b. Internal Flow

Apply Continuity

$$U_1 w_1 = U_2 (w_2 - 2\delta) + 2 \int_0^\delta U \partial y$$

Solve for $U_2$ - integrated form of continuity.

Possible Closures:

1. $-u'v' = \ell^2 \left[ \frac{\partial \bar{u}}{\partial y} \frac{\partial \bar{u}}{\partial y} \right]$

2. $u'v' = \nu \frac{\partial \bar{u}}{\partial y}$

But we will pursue a different approach.

Basic method of solution.

Momentum integral technique

1. Integrate $x$-mom and continuity across BL, this will convert 2 PDE’s to one ODE in terms of new variables.
2. Select an auxiliary equation. Empirically or analytically based; 2$^{\text{nd}}$ relation.
3. Select a skin friction relationship (empirically or analytically based) 3rd relation.
4. Select a velocity relationship with as wide of a family of profiles as appropriate.

Simple

Complex

This approach yields information on $\bar{u}(x, y)$ and $\tau_\omega(x)$ but not $u'v'$ behavior.

Integral Equation Approach

Let $\tau = \mu \frac{\partial U}{\partial y} - \rho u'v'$ the total shear stress.

1. $U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = U_\infty \frac{\partial U_\infty}{\partial x} + \frac{1}{\rho} \frac{\partial \tau}{\partial y}$

B.C. at $y = 0$, $U = V = 0$

$y = \delta$, $U = U_\infty(x)$

Integrate continuity first.

$$\frac{\partial V}{\partial y} = -\frac{\partial U}{\partial x}$$

$$\int_0^y \frac{\partial V}{\partial y} dy = -\int_0^y \frac{\partial U}{\partial x} dy$$

$$\frac{\partial}{\partial y} \int_0^y V dy = V(y) - V(0)$$

Then $V = -\int_0^y \frac{\partial U}{\partial x} dy + f(x)$

$f(x) = 0$ because $V(x, 0) = 0$

Substitute this expression for $V$ into $x$-momentum and integrate across B-L.
3. \[
\int_0^\delta \frac{\partial U}{\partial x} \, dy - \int_0^\delta \frac{\partial U}{\partial y} \left( \int_0^\delta \frac{\partial U}{\partial x} \, dy \right) \, dy = \int_0^\delta \frac{\partial U}{\partial x} \, dx + \frac{1}{\rho} \int_0^\delta \frac{\partial \tau}{\partial y} \, dy
\]

Last term in 3.
\[
\frac{1}{\rho} \int_0^\delta \frac{\partial \tau}{\partial y} \, dy = \frac{1}{\rho} \left[ \tau(\delta) - \tau(0) \right] = -\frac{\tau_w}{\rho}
\]
\[
\mu \int_0^\delta \frac{\partial f}{\partial y} \, dy = -\rho \frac{\tau_w}{\rho}
\]

Let \( I \equiv \int_0^\delta m \, dn = \left[ \int_0^\delta m \right]_0^n \)
\[
m = \int_0^\delta \frac{\partial U}{\partial x} \, dy \quad dn = \frac{\partial U}{\partial y} \, dy
\]
\[
dm = \frac{\partial}{\partial y} \left( \int_0^\delta \frac{\partial U}{\partial x} \, dy \right) \, dy \quad n = U
\]
\[
dm = \frac{dm}{dx} \, dx + \frac{dm}{dy} \, dy
\]

Since integrating with respect to \( y \) at fixed \( x \).
\[
dm = \frac{\partial}{\partial x} \left( \int U \, dy \right) \, dy = \frac{\partial U}{\partial x} \, dy
\]
\[
I = U \int_0^\delta \frac{\partial U}{\partial x} \, dy \left|_0^\delta \right. - \left. \int_0^\delta U \frac{\partial U}{\partial x} \, dy \right.
\]
\[
= U \int_0^\delta \frac{\partial U}{\partial x} \, dy - \left[ \int_0^\delta U \frac{\partial U}{\partial x} \, dy \right]
\]

Substitute into the integrated form of 3.

4. \[
2 \int_0^\delta \frac{\partial U}{\partial x} \, dy - U_\infty \int_0^\delta \frac{\partial U}{\partial x} \, dy - \int_0^\delta \frac{\partial U}{\partial x} \, dy = -\frac{\tau_w}{\rho}
\]

Now change signs on 4 and add and subtract \( \int_0^\delta \frac{\partial U}{\partial x} \, dy \).
Result

\[ \int_0^\delta \frac{\partial}{\partial x} \left[ U \left( U_\infty - U \right) \right] dy + \frac{\partial U_\infty}{\partial x} \int_0^\delta \left( U_\infty - U \right) dy = \frac{\tau_w}{\rho} \]

Normalized (non-dimensionalize) by \( \frac{1}{U^2} \).

5. \[ \frac{1}{U_\infty^2} \frac{\partial}{\partial x} \left[ U_\infty^2 \int_0^\delta \frac{U}{U_\infty} \left( 1 - \frac{U}{U_\infty} \right) dy \right] + \frac{1}{U_\infty} \frac{\partial U_\infty}{\partial x} \int_0^\delta \left( 1 - \frac{U}{U_\infty} \right) dy = \frac{\tau_w}{\rho U_\infty^3} \]

Define

\[ \delta^* \equiv \int_0^\delta \left( 1 - \frac{U}{U_\infty} \right) dy \] Displacement thickness

\[ \theta \equiv \int_0^\delta \frac{U}{U_\infty} \left( 1 - \frac{U}{U_\infty} \right) dy \] Momentum thickness

\[ H_{1z} \equiv \frac{\delta^*}{\theta} \] Shape factor

\[ C_f \equiv \frac{\tau_w}{\frac{1}{2} \rho U_\infty^2} \] Skin friction coefficient

Rewrite 5 with new definition

\[ \frac{1}{U_\infty^2} \frac{\partial}{\partial x} \left( U_\infty^2 \theta \right) + \frac{1}{U_\infty} \frac{\partial U_\infty}{\partial x} \delta^* = \frac{C_f}{2} \]

Now expand

\[ \frac{\partial \theta}{\partial x} + \frac{\theta}{2 U_\infty} 2 \frac{\partial U_\infty}{\partial x} + \frac{1}{U_\infty} \frac{\partial U_\infty}{\partial x} \delta^* = \frac{C_f}{2} \]

Combine 2 middle terms
\[
\frac{1}{U} \frac{\partial U}{\partial x} \left( \delta^* + 2\theta \right) \frac{1}{(H_{12} + 2)\theta}
\]

Final Result:

\[
\frac{\partial \theta}{\partial x} = \frac{C_f}{2} - \left( H_{12} + 2 \right) \frac{\theta}{U_{\infty}} \frac{\partial U_{\infty}}{\partial x}
\]

ODE to be solved for \( \theta(x) \), \( C_f(x) \), \( H_{12}(x) \)

New values replacing \( \bar{u}, \bar{v} \).