The Effect of Channel Length on the Residual Circulation in Tidally Dominated Channels

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(Manuscript received 25 March 2004, in final form 21 March 2005)

ABSTRACT

With an analytic model, this paper describes the subtidal circulation in tidally dominated channels of different lengths, with arbitrary lateral depth variations. The focus is on an important parameter associated with the reversal of the exchange flows. This parameter ($\delta$) is defined as the ratio between the channel length and one-quarter of the tidal wavelength, which is determined by water depth and tidal frequency. In this study, a standard bottom drag coefficient, $C_D = 0.0025$, is used. For a channel with $\delta$ smaller than 0.6–0.7 (short channels), the exchange flow at the open end has an inward transport in deep water and an outward transport in shallow water. This situation is just the opposite of channels with a $\delta$ value larger than 0.6–0.7 (long channels). For a channel with a $\delta$ value of about 0.35–0.5, the exchange flow at the open end reaches the maximum of a short channel. For a channel with a $\delta$ value of about 0.85–1.0, the exchange flow at the open end reaches the maximum of a long channel, with the inward flux of water occurring over the shoal area and the outward flow in the deep-water area. However, near the closed end of a long channel, the exchange flow changes direction along the channel from the head to the open end of the channel. For a channel with a $\delta$ value of about 0.6–0.7, the tidally induced subtidal exchange flow at the open end reaches its minimum when there is little flow across the open end and the water residence time reaches its maximum. The mean sea level increases toward the closed end for all $\delta$ values. However, the spatial gradient of the mean sea level in a short channel is much smaller than that of a long channel. The differences between short and long channels are caused by a shift in dynamical balance of momentum or, equivalently, a change in tidal wave characteristics from a progressive wave to a standing wave.

1. Introduction

Tides behave differently in long and short channels. In a long channel, the tidal wave is progressive because of a significant damping effect of bottom friction; in a short channel, it is a standing wave because of a relatively smaller damping effect. Note that the length of the channel is not measured by the geometric length alone but by a ratio between the geometric length and the tidal wavelength. With a given bottom drag coefficient, this ratio determines the overall damping effect along the channel. For convenience of discussion, we will use a nondimensional length parameter $\delta$ defined by the ratio between the geometric length of the channel and one-quarter of the wavelength:

$$\delta = \frac{4L}{\lambda} \left( \lambda = \frac{\sqrt{gh}}{f} \right),$$

where $g$, $h$, and $f$ are the gravitational acceleration, depth, and tidal frequency, respectively.

The rationale for the selection of this parameter is based on the following considerations: 1) For a frictionless tidal wave propagating in a channel, resonance occurs when the channel length equals one-quarter of the wavelength, or when $\delta = 1$. With friction, the magnitude of the response is finite, and the peak is shifted...
simplification in producing the second-order residual circulation? The work presented in this paper will revisit the model and solve the problem without implementing the crude approximation used in the earlier model. The first-order solution for tide is then used to solve the second-order mean circulation by a normal perturbation method. This approach does not pose any technical challenge for the first-order tide by using the solution of Li and Valle-Levinson (1999), nor does it significantly complicate the solution of the second order. The improvement of the present model is made in the first-order tidal flow, and it is dependent on the depth variations. The results provide some insight to the variation of flow regimes among channels of different lengths and within different segments of the same channel for long channels. The results at the mouth of the channel, where the exchange flow pattern is of most interest, are also consistent with the earlier model for long channels in which the tidal wave is progressive. The results for short channels or near the closed end of long channels are different from the earlier model. This result indicates that the residual flow regime changes with channel length or varies within a long channel.

2. The model and solutions

The basic idea of the analytic model that we are using is to solve the first-order \( O(\epsilon) \) equations of the tide first and then to use it as the forcing for the second-order tidally averaged motion. By varying relevant parameters, the response of the latter can be analyzed. In mathematical terms, this is a normal perturbation method.

We are only interested in narrow channels (a few kilometers) open at one end (where it is connected to the ocean) and with no density gradients, in which Coriolis effect can be neglected. A practical definition of a narrow channel is one in which the lateral variation of the tidal amplitude \( (\Delta\zeta_0) \) is much smaller than the cross-channel mean of the tidal amplitude \( (\bar{\zeta}_0) \)—that is, \( \Delta\zeta_0 \ll \bar{\zeta}_0 \) (Li and Valle-Levinson 1999).

In the model, the \( x \) axis is along the boundary and points toward the closed end of the channel. The \( y \) axis is along the open boundary at \( x = 0 \). A single-frequency, semidiurnal tide is imposed at the open end of the channel, and both the amplitude and phase of the sea level variation at the open end are assumed to be uniform across the channel.

The depth-averaged, shallow-water momentum and continuity equations are used:
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -g \frac{\partial \zeta}{\partial x} - \frac{\beta}{h} u + \frac{\beta}{h^2} u \zeta,
\]
\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -g \frac{\partial \zeta}{\partial y} - \frac{\beta}{h} v + \frac{\beta}{h^2} v \zeta,
\]
\[
\frac{\partial \zeta}{\partial t} + \frac{\partial (h + \zeta)u}{\partial x} + \frac{\partial (h + \zeta)v}{\partial y} = 0,
\]

where \( u, v, \zeta, h, x, y, t, C_D, \) and \( g \) are longitudinal velocity, lateral velocity, elevation, undisturbed water depth, longitudinal coordinate, lateral coordinate, time, drag coefficient, and the gravitational acceleration, respectively.

The longitudinal velocity at the head and the lateral velocity at the side boundaries are zero. The boundary conditions for the model can be expressed as
\[
u|_{x=L} = 0, \quad \nu|_{y=0,D} = 0, \quad \text{and} \quad \zeta|_{x=0} = \text{Re}(\zeta_0e^{i\omega t}),
\]
where the constants \( D, \sigma, \) and \( \zeta_0 \) are the channel width, tidal frequency, and amplitude at the mouth, respectively, and \( i = \sqrt{-1}. \) Here, \( \text{Re} \) is an operator that takes the real part of the function within the parentheses. Mass balance requires that the cross-channel integration of the tidally averaged transport over a tidal cycle must be zero under idealized conditions (i.e., when there is only one tidal frequency and there is no change in mean sea level from low-frequency motions such as remote wind effect), and we have
\[
\int_0^D \frac{\partial}{\partial y} (h + \zeta u) \, dy = 0.
\]

It can be seen later that this condition is only useful for the second-order \( [O(\epsilon^2)] \) tidally averaged residual circulation.

As shown in Li and O’Donnell (1997), a Fourier decomposition (Proudman 1953; Parker 1984) of the quadratic friction leads to the following equations, correct to the second order:
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -g \frac{\partial \zeta}{\partial x} - \frac{\beta}{h} u + \frac{\beta}{h^2} u \zeta,
\]
\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -g \frac{\partial \zeta}{\partial y} - \frac{\beta}{h} v + \frac{\beta}{h^2} v \zeta,
\]
\[
\frac{\partial \zeta}{\partial t} + \frac{\partial (h + \zeta)u}{\partial x} + \frac{\partial (h + \zeta)v}{\partial y} = 0,
\]
in which
\[
\beta = \frac{8C_D U_0}{3\pi},
\]
where \( C_D \) and \( U_0 \) are the bottom drag coefficient and the magnitude of the longitudinal velocity, respectively. In this study, \( U_0 \) is chosen to be a constant \((0.83 \text{ m s}^{-1})\). In (5) the quadratic friction has been decomposed into a linear part and a higher-order term. The advantage of (5) over the original shallow-water equations is that a formal perturbation method can be readily applied for an analytical solution.

To be specific, let the solution be a superposition of the first-order, second-order, and higher-order constituents; that is,
\[
u = \nu_1 + \nu_2 + \cdots, \quad \text{and} \quad \zeta = \zeta_1 + \zeta_2 + \cdots.
\]

It is assumed here that the first-order tide, denoted by the subscript 1, has a magnitude proportional to \( \epsilon = \zeta_0/h \) and the second-order quantity, denoted by the subscript 2, has a magnitude proportional to \( \epsilon^2 \). In many estuaries and tidal channels, \( \zeta_0 \) is small relative to the mean water depth. A typical \( \epsilon \) is 0.1. After substituting the above equations into the momentum and continuity equations, one can obtain the first-order equations by neglecting terms that are of second order or smaller. The second-order equations can then be obtained by neglecting terms of third order or smaller. If the first-order linear equations can be solved, the second-order equations may also be tractable because they are linear equations driven by products of the known first-order solution. A strict and standard approach is to nondimensionalize the equations and separate equations of different orders by neglecting higher-order terms. The procedure is described in the appendix. Here we will use the dimensional equations solely for the purpose of presentation. The dimensional equations are the exact conversions of their nondimensional counterparts.

The first-order linearized equations of (5) are (see the appendix for the nondimensional version)
\[
\frac{\partial \nu_1}{\partial t} = -g \frac{\partial \zeta_1}{\partial x} - \frac{\beta}{h} \nu_1,
\]
\[
\frac{\partial \nu_1}{\partial t} = -g \frac{\partial \zeta_1}{\partial y} - \frac{\beta}{h} \nu_1, \quad \text{and} \quad \frac{\partial \zeta_1}{\partial t} + \frac{\partial (h + \zeta_1)\nu_1}{\partial x} + \frac{\partial (h + \zeta_1)\nu_1}{\partial y} = 0.
\]

The depth is assumed to be a sufficiently smooth (differentiable) but otherwise arbitrary function of the cross-channel position:
\[
h = h(y) \quad (0 < y < D; \ D \text{ is the channel width}).
\]
For a single-frequency tide, the solution can be expressed as

\[ u_1 = \text{Re}(U e^{i\sigma t}), \quad v_1 = \text{Re}(V e^{i\sigma t}), \quad \zeta_1 = \text{Re}(A e^{i\sigma t}), \]

(10)

where \( \sigma, i, U, V, \) and \( A \) are the angular frequency of the tide, the unit imaginary number \( \sqrt{-1} \), the complex amplitude of the longitudinal velocity, the complex amplitude of the lateral velocity, and the complex amplitude of the tidal elevation, respectively. The boundary conditions are

\[ U|_{y=L} = 0, \quad V|_{y=0,D} = 0, \quad \text{and} \quad A|_{x=0} = \zeta_0. \]

(11)

A solution of (8) for the velocity and tidal elevation has been obtained by Li and Valle-Levinson (1999) with an assumption that the cross-channel variation of surface elevation is much smaller than that of the tidal amplitude. In complex form, the elevation is

\[ A = \zeta_0 \cos[\omega(x - L)] / (\omega L), \]

(12)

in which

\[ \omega^2 = \frac{i \sigma D}{f}, \quad \text{with} \quad f = -\int_{0}^{D} \frac{gh}{\sigma + \beta h} dy, \]

(13)

and the along-channel velocity is

\[ U = \frac{g}{i \sigma + \beta h} \frac{\omega}{\cos(\omega L)} \sin[\omega(x - L)]. \]

(14)

Using the continuity equation, the lateral velocity can be obtained as

\[ V = \frac{1}{h} \left( i \sigma y + \int_{0}^{D} \frac{gh \omega^2}{\sigma + \beta h} dy \right) A. \]

(15)

Equations (10) and (12)–(15) constitute the solution for the first-order equations in (8), satisfying the boundary conditions in (11). The difference between the results of (12)–(15) and those of Li and O’Donnell (1997) is that the latter use a solution of a flat-bottom channel for the first-order tide while the former allows an arbitrary cross-channel variation in water depth—that is, \( h = h(y) \). The condition for this new solution is that the cross-channel difference in tidal amplitude (\( \Delta \zeta_{0|c} \)) is much smaller than the cross-channel mean of the tidal amplitude (\( \zeta_0 \)); that is, \( \Delta \zeta_{0|c}/\zeta_0 \approx 1 \). The quantity \( \Delta \zeta_{0|c}/\zeta_0 \) is also an estimate of the error of the first-order model. In the James River estuary, for example, the maximum \( \Delta \zeta_{0|c}/\zeta_0 \) is about 5% (Li and Valle-Levinson 1999). The first-order solution can also allow a variable drag coefficient as shown by Li et al. (2004). In this paper, however, we will assume that \( C_D \) and \( \beta \) are constant for convenience.

Similar to Li and O’Donnell (1997), the tidally averaged second-order equations can be obtained as (see the appendix for the nondimensional version)

\[ \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = -g \frac{\partial \zeta_2}{\partial x} - \frac{\beta \bar{V}}{h} + \frac{\beta}{h^2} \frac{\partial \zeta_1}{\partial y}, \]

\[ \frac{\partial v_1}{\partial x} + \frac{\partial u_1}{\partial y} = -g \frac{\partial \zeta_2}{\partial y} - \frac{\beta \bar{V}}{h} + \frac{\beta}{h^2} \frac{\partial \zeta_1}{\partial x}, \]

\[ \frac{\partial \zeta_2}{\partial x} + \frac{\partial \bar{V}}{\partial y} + \frac{\partial \bar{V}}{\partial y} + \frac{\partial \zeta_1}{\partial y} = 0, \]

(16)

in which the overbars indicate temporal averages over a complete tidal cycle.

To solve the tidally averaged second-order equations in (16), we define the transport velocity as in Robinson (1983) and Li (1996):

\[ v_T = (1 + \zeta \delta) \bar{v} = \bar{v} + \frac{\zeta \bar{v}}{h} + \text{higher-order terms}, \]

(17)

in which the boldface \( \bar{v} \)s are velocity vectors—for example, \( \bar{v}_1 = (u_1, v_1) \). Because the total volume flux of water across unit width at a given time is \( (h + \zeta) \bar{v} \), the transport velocity defined by (17) is a quantity that has the dimension of velocity and yields the net flux across unit width when multiplied by the undisturbed water depth \( h \). With this definition, the equations in (16) can be rearranged to

\[ u_T = \frac{2}{h} \frac{\zeta_1 u_1}{h} - \frac{h}{\beta} \left( u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} \right) - \frac{gh \Delta \zeta_2}{\beta \partial x}, \]

\[ v_T = \frac{2}{h} \frac{\zeta_1 v_1}{h} - \frac{h}{\beta} \left( u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} \right) - \frac{gh \Delta \zeta_2}{\beta \partial y}, \]

and

\[ h \frac{\partial u_T}{\partial x} + \frac{\partial \bar{V}}{\partial y} = 0. \]

(18)

Note that 1) for a narrow channel the surface elevation of the water is almost uniform along the same cross-channel transect (Friedrichs and Hamrick 1996; Li 1996; Li and O’Donnell 1997) such that \( \Delta \zeta_2/\partial x \) is independent of cross-channel coordinate \( y \), 2) no-normal-flow condition applies on the lateral boundaries (i.e., \( \bar{v}_1 = 0 \) at \( y = 0 \) and \( D \)), and 3) the integrated total transport due to tide is zero under idealized conditions (i.e., when there is no change in mean sea level from
low-frequency motions such as remote wind effect; thus
\[ \int_0^D h u_T \, dy = 0, \quad (19) \]
which is a different format of (4). Using these conditions and by integrating the third equation of (18) across the channel we can solve the tidally averaged pressure gradient to be proportional to
\[ \frac{\partial \xi_2}{\partial x} = - \frac{1}{g} \int_0^D h^2 \, dy \times \left[ 2\beta \int_0^D \xi_1 u_1 \, dy - \int_0^D h^2 \left( \frac{\partial u_1}{\partial x} + v_1 \frac{\partial h}{\partial y} \right) \, dy \right]. \quad (20) \]
From (20) the tidally averaged pressure gradient can be solved with the first-order solution \( u_1, v_1, \) and \( \xi_1 \) and an arbitrary depth function \( h(y) \). The restriction to \( h(y) \) is that it must be smooth and nonzero at all positions. By substituting the tidally averaged surface slope into the first equation of (18), the along-channel component of the transport velocity can be calculated. The lateral component of the transport velocity cannot be resolved from the second equation of (18) because the mean lateral pressure gradient is unknown. The cross-channel component of the transport velocity, however, can be resolved from the mass-conservation requirement by lateral integration of the third equation of (18) as demonstrated in Li and O’Donnell (1997):
\[ v_T = -\frac{1}{h} \frac{\partial}{\partial x} \left( \int_0^D h u_T \, dy \right). \quad (21) \]
In the following section, we will analyze and discuss the above solution with different channel lengths and a given depth function.

### 3. Computations and results

In the calculations of the above solution, we use a semidiurnal tidal period (\( T = 12 \) h) with a standard bottom drag coefficient \( C_D = 0.0025 \) (Proudmman 1953), which yields \( \beta = 0.00176 \) m s \(^{-1}\) given \( U_0 = 0.83 \) m s \(^{-1}\). To examine the effect of channel length on the semidiurnal induced mean flow field and resulting exchange rate under different bathymetry conditions, we choose the length ratio of the model, \( \delta \) as defined by (1), to vary between 0.052 and 1.56 (or 5 and 150 km for a semidiurnal tide), with a constant width of 2 km. The change of width has been found to have little effect on the structure of the semidiurnal induced mean flow (Li 1996; Li and O’Donnell 1997) unless the width is too large such that the lateral variation of the surface elevation is comparable to the tidal amplitude. We will therefore focus on the effects of channel-length variations only. We have experimented with several different depth functions including those used in Li and Valle-Levinson (1999). For the presentation here we will only discuss the results from one depth function representative of the characteristics of the solution. The selected cross-channel depth function (Fig. 1b) is defined by the following equation (all parameters are in meters):
\[ h(y) = h_1 + h_2 e^{-\left(y - D/2 \right)^2/\delta^2}. \quad (22) \]
In the above equations, \( h_1 = 5 \) m and \( h_2 = 10 \) m. This depth function is symmetric about the axis of the channel. Such a depth function, though idealized, is representative of the main features of many coastal plain estuaries: a deep channel and shallow shoals, all on the order of a few to tens of meters.

To calculate the mean surface slope from (20), we need to calculate the tidally averaged function of the product of two functions, for example, \( \xi_1 u_1 \). Assume that \( a = \text{Re}(Ae^{i\omega t}) \) and \( b = \text{Re}(Be^{i\omega t}) \), and if we use subscripts \( R \) and \( I \) to represent the real and imaginary parts, respectively, that is,
\[ A = A_R + iA_I \quad \text{and} \quad B = B_R + iB_I. \quad (23) \]
then it can be proven that

$$\bar{a}b = \text{Re}(A e^{i\omega t})\text{Re}(B e^{i\omega t}) = (A_R B_R + A_I B_I)/2.$$  (24)

For convenience, the first-order solution—that is, $u_1$, $v_1$, and $\zeta_1$—is calculated on equally spaced nodes. Because the solution is analytical, the grid size $dx$ and $dy$ can be arbitrary. The calculation of the first-order spatial derivatives required for the solution of the subtidal fields uses the sixth-order compact scheme of Lele (1992) for the interior nodes and the sixth-order scheme of Carpenter et al. (1993) for the near boundary nodes. The integrations such as those in (13) and (20) are calculated using the Simpson method.

In the calculations, we use a 5-km step to vary the length of the channel; that is, the length of the channel is chosen to be 5, 10, 15, . . . , 150 km, respectively. In doing so, for the given depth function, we calculate the tidal and mean flow fields with 30 different channel lengths. For each of these calculations, we choose 51 cross-channel positions and 61 along-channel positions for the flow fields and all related variables. By examining the results, the change of the flow patterns with the channel length can be assessed. In the following, we will examine the exchange flow patterns from two perspectives: 1) selected locations and 2) plane view of the entire flow field.

a. Residual circulation

1) Exchange flow at selected locations

By selecting different positions across the channel, the response of the subtidal flow in the along-channel direction at the open end as a function of the channel length is shown in Fig. 1a. For all of the depth functions with which we experimented [results from only one depth function defined by (22) are discussed for brevity], the tidally induced mean flow for “short” channels in the deep water is positive (into the channel or “landward”) while the mean flow in the shoals is negative (out of the channel or “seaward”). This flow pattern is opposite of that of the model results of Li and O’Donnell (1997), which are applicable only to a “long channel” with a progressive tidal wave. In the model of Li and O’Donnell (1997), the effect of the lateral variability of the first-order tidal flow due to the variation in depth was neglected for convenience, which makes the model inappropriate for short channels. The results in this paper show that, as the length of the channel increases, the magnitude of the exchange flow reaches its maximum when $\delta$ is at about 0.42 and then decreases and reaches its minimum (0.0) when the length ratio $\delta$ is about 0.64 for the selected depth function. For the selected depth function, the corresponding length scales are listed in Table 1. When the channel length is further increased, the exchange flow at the open end reverses directions: the flow in the deep center is outward while the flow in the shoal is inward, which is consistent with the model result of Li and O’Donnell (1997). When the channel length ratio is about 0.92, the mean flow reaches its maximum again at the open end.

Table 1. Mean depth ($h_0$) and length scales.

<table>
<thead>
<tr>
<th>$h_0$ (m)</th>
<th>$\lambda/4$ (km)</th>
<th>$0.6 \times \lambda/4$ (km)</th>
<th>$L = 30$ km</th>
<th>$L = 50$ km</th>
<th>$L = 100$ km</th>
<th>$L = 105$ km</th>
<th>$L = 150$ km</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.1</td>
<td>96</td>
<td>58</td>
<td>$\delta = 0.31$</td>
<td>$\delta = 0.52$</td>
<td>$\delta = 1.04$</td>
<td>$\delta = 1.09$</td>
<td>$\delta = 1.56$</td>
</tr>
</tbody>
</table>

All of the experiments have shown that, regardless of the actual depth distributions across the channel, the exchange flows are consistent qualitatively among all short channels and among all long channels. Here the division between short and long channels is made by $\delta = 0.6-0.7$ (0.64 for the example presented). The maximum exchange flow for a short channel occurs when the length ratio is about 0.35-0.5 (0.42 for the example presented). The maximum exchange flow for a long channel occurs when the length ratio is about 0.85-1.0 (0.92 for the example presented). The variabilty of these length scales is due to the fact that overall frictional effect can be different for models with different depth functions even for the same drag coefficient. The experiments also demonstrate that the mean flow pattern can be altered if the water-depth distribution across the channel is altered. An example of this is a dredging activity or storm event that can deepen a channel, modifying and possibly reversing the exchange flow patterns if the change in depth causes sufficient variations in $\delta$.

2) Residual flow field

As a representative example, the plane view of the residual flow field in a long channel with the depth function defined in (22) is shown in Fig. 2. The length of the channel in this case is 105 km, and the corresponding length ratio $\delta$ is about 1.09 (Table 1). For brevity, we have omitted the same plot for a short channel because the flow pattern in a short channel is the same as that in the closed end of a long channel. This fact is verified by all of the experiments with different depth functions that we have used. The residual flow field in this case (Fig. 2) is symmetric about the axis of the
channel because the cross-channel depth distribution is symmetric and Coriolis force is not considered in this study. The most interesting feature of this figure is that the tidally induced residual flow changes its pattern from a short channel to a long channel or from the closed end to the open end. Near the closed end (δ between 0.5 and 1.0 or x between 50 and 100 km), the flow pattern is such that the mean inward flow occurs in the deep water and the mean outward flow occurs in the shallow water. After a short transition zone (x around 40 km), the flow pattern switches to an opposite sense such that the mean inward flow occurs in the shallow water and the mean outward flow occurs in the deep water. The present result for short channels is different from that of Li and O’Donnell (1997), which uses a crude approximation for the first-order tide. In section 4, we will further discuss the change of flow regimes from short to long channels and from the closed end to the open end in a long channel.

b. Exchange rate and residence time

The along-channel component of the mean flow can be either positive (inward) or negative (outward). By integrating all positive volume flux per unit time across a given section of the channel, the magnitude of the rate of exchange at that section can be examined. If we integrate all negative volume flux per unit time instead, we will get the same exchange rate with a negative sign. The exchange rate can then be used to calculate the residence time as the ratio between the volume of the channel and the exchange rate. In the present study, for given depth function and parameters, the length of the channel determines the exchange rate and residence time. Figure 3 shows the response of the exchange rate (Fig. 3a) and residence time (Fig. 3b) to the change of channel length for the selected depth function (22). For all depth functions and parameters with which we have experimented, the exchange rate peaks at δ ~ 0.35–0.5 (for short channels) and δ ~ 0.85–1.0 (for long channels). The minimum exchange rate occurs at δ ~ 0.6–0.7, corresponding to the minimum residual flow at the open end (Figs. 1a and 2). The residence time remains small and relatively constant for the short channels until δ ~ 0.6–0.7 when it reaches maximum and as the exchange rate reaches its maximum. After this peak, the residence time decreases rapidly and then increases again gradually with the channel length.

4. Discussion

a. Momentum and mass balance

The solutions presented here require an assumption that the lateral variation of the surface elevation is small in a narrow channel. Although this assumption has been supported by observations using pressure sensors across the James River estuary (Li and Valle-Levinson 1999), a check on the momentum and mass balances may quantify the errors and verify that the results do satisfy the momentum and continuity equations with reasonably small residuals or errors. For this purpose, we have calculated the errors of the momentum and continuity equations for all of the experiments. These errors are examined at both the first order for the tidal motion and higher order with the full nonlinear equations for the residual solution.

b. Exchange rate and residence time

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The errors of both momentum and continuity equations are calculated at every computational grid point with velocity fields along and across the channel. It shows that the momentum equation is accurately satisfied within the computational error of the computer. This is because the along-channel velocity is solved exactly using the $x$ momentum equation and the cross-channel velocity is obtained exactly by integrating the continuity equation after the surface elevation is solved. This can also be verified by substituting the solution into the original equations. The calculation of the errors of the equations serves the purpose of verifying that the computer programs used in the calculations are correct. The surface elevation, on the other hand, is solved with the approximation that the elevation is $y$ independent. As a result, the largest residual is seen in the continuity equation, which has a maximum relative error of about $2 \times 10^{-3}$ or 0.2%, a small value nevertheless.

The above examination is made to the momentum and continuity equations. To test the mass conservation further, we have also directly calculated the net mass flux through different cross sections along the channel. The net mass flux is obtained by integrating the along-channel component of the transport velocity ($u_T$) across the channel. All experiments show practically zero net mass flux. Therefore, the solution does satisfy the equations, and the mass is conserved accurately.

**b. Residual transport velocity**

To help to interpret the switch of exchange flow patterns from short to long channels, we now analyze the residual transport velocity (18). We write the right-hand side of (18) as a sum of three terms:

$$u_T = u_{T1} + u_{T2} + u_{T3},$$

in which

$$u_{T1} = 2 \frac{\xi u_1}{h}, \quad u_{T2} = \left( \frac{h}{\beta} \left( \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) \right),$$

and

$$u_{T3} = - \frac{gh \partial \xi}{\beta}.$$

The first term $u_{T1}$ is a combination of two mechanisms: the bottom friction and the finite tidal amplitude; each is equal to $2 h \xi u_1/h$, and thus the total has a factor of 2. The second term $u_{T2}$ is a result of advection. The third term $u_{T3}$ is caused by the mean pressure gradient force induced by a subtidal surface slope. Among these terms, the first two are directly determined by the first-order tidal motion. The last term $u_{T3}$ is proportional to the mean surface slope of the water along the channel, which can also be calculated by the first-order tide through (20). By integrating the mean surface slope along the channel, the mean surface elevation is obtained. Figure 4 shows the results of this integration for all model runs with the given depth function and parameters. Each curve represents the along-channel distribution of the mean surface for one model run (i.e., with a certain channel length). The results show that the mean surface always increases toward the head of the channel for all cases. This produces an outward-directed (negative) pressure gradient force. The surface slope is strongly dependent on the length of the channel. For a very short channel, the slope is very small—almost flat (the short lines of Fig. 4). As the length increases, the slope increases and then oscillates but with a decreasing magnitude of oscillation in a manner similar to the response of tidal velocity as the length of the channel increases (see figures in Li 2001).

The increase of mean surface elevation toward the head of the channel can be explained by the propagation of tidal waves. We know that tide in a semienclosed channel can be considered as an incident wave plus a reflected wave [which can be seen by expressing (9) and (11) in real instead of complex format]. If there is no bottom friction, the incident and reflected waves will have equal amplitudes and are out of phase (e.g., Defant 1961). With bottom friction, the wave energy is attenuated as it propagates. For a long channel, the bottom friction causes more decrease in the reflected wave than in the incident wave because at a given location the reflected wave has bounced from the head and thus traveled longer distance than the incident wave. The net result is a stronger incident wave and weaker re-
reflected wave, especially at the open end of the channel. The stronger the incident wave is, the closer the wave is to a progressive wave. A pure progressive wave only exists in an infinitely long channel (regardless of whether there is friction) in which there is no reflection of wave energy such that the elevation and flow are in phase (with 0° phase difference). Equivalently, the high tide of a progressive wave has a maximum inward flow (or flood). The low tide has a maximum outward flow (or ebb). The maximum flood velocity is approximately the same as the maximum ebb velocity in magnitude for a first-order approximation. Because at high tide the total water depth is larger, this progressive wave causes a net inward transport of water mass, which results in an increased mean sea level toward the head. The longer the channel is, the more pronounced effect the bottom friction will have. For short channels, the phase difference between elevation and flow is close to 90° and the transport due to wave motion is nearly zero, causing a “flat” mean surface. The mean surface slope, whenever it exists, causes a seaward-directed pressure gradient force that drives a net outward flow, which in turn prohibits the mean sea level slope from further increase; a state of balance is thus reached, which is the solution.

1) SHORT CHANNEL

In this section, we will examine the contributions of each of the three terms in (25). We will discuss the distributions of each term for a short channel and a long channel, respectively. For a short channel, \( u_{T1} \) is positive (into the channel) in the deeper central part and is negative in the shoal area (Fig. 5a). This is because tide in a short channel is closer to a standing wave. A pure standing wave is such that the elevation \( \zeta \) and the along-channel velocity have a 90° phase difference. As a result, the tidally averaged product of \( \zeta \) and \( u \)—that is, \( \bar{\zeta u} \)—is zero. With bottom friction, \( \bar{\zeta u} \) is not exactly zero but can go either way (positive or negative) for a short channel. To be more specific, the tidally averaged \( \bar{\zeta u} \) or \( u_{T1} \) is proportional to \( \sin(\alpha t) \cos(\alpha t + \alpha) \), in which \( \alpha \) is the phase difference in excess of 90° between the surface elevation \( \zeta \) and the along-channel velocity \( u \) [note that \( \sin(\alpha t) \) and \( \cos(\alpha t) \) have a 90° phase difference]. This quantity is a function of cross-channel distance or water depth and is identically zero if there is no friction when the phase difference is exactly 90° (i.e., \( \alpha = 0 \)). When there is bottom friction, \( \alpha \) is zero only at a certain value of \( h = \bar{h} \), which should be around the mean depth. Because bottom friction causes the along-channel velocity in shallower water to lead that in deeper water, that is, \( \alpha \) in shallower water is larger than that in deeper water (Li 2001), it follows that for \( h \) greater than \( \bar{h} \), \( \alpha \) is less than 0 and for \( h \) less than \( \bar{h} \) \( \alpha \) is greater than 0. Because

\[
\sin(\alpha t) \cos(\alpha t + \alpha) = - \sin(\alpha) \sin^2(\alpha t) = - \frac{\sin(\alpha)}{2},
\]

(27)
the tidally averaged $\zeta u$ is an odd function of $\alpha$, which leads to

$$u_{T1} \approx \begin{cases} >0 & (h > \tilde{h}) \\ <0 & (h < \tilde{h}) \end{cases}.$$

This qualitatively explains the negative values of $u_{T1}$ in the shoal area and positive values in the deep water for short channels (Fig. 5a). The mean depth for the given depth function is about 8.1 m. The zero $u_{T1}$ for this case occurs at $y = 0.65$ and 1.35 km, corresponding to a depth $\tilde{h}$ of about 8 m. This explains the flow pattern of Fig. 5a.

The second term $u_{T2}$ is all positive, which indicates that advection induces inward transport of water mass. The deeper the water is, the larger this inward transport is (Fig. 5b). The third term $u_{T3}$ is all negative (Fig. 5c), which is consistent with the positive mean surface slope as discussed earlier (Fig. 4). A larger outward transport by $u_{T3}$ is through the deep water. This is because the pressure gradient is laterally independent and thus the vertically integrated transport due to this mechanism is proportional to water depth, as is evident from the definition of $u_{T3}$ [(26)]. It turns out that although $u_{T1}$ has a smaller magnitude than either $u_{T2}$ or $u_{T3}$, the inward advective transport and the outward transport due to the mean pressure gradient force are similar in magnitude. The net effect, or the sum of all the three terms $u_T$ (Fig. 5d), is therefore similar to $u_{T1}$, which is a result of the finite amplitude of the tide and the phase-difference distribution in different water depth due to bottom friction. Therefore, the net transport in a short channel is such that the inward flow occurs through the deep water while the outward flow occurs through the shallow water.

2) LONG CHANNEL

For a long channel, the tidally averaged product of surface elevation and velocity—that is, $\zeta u$—can be expressed as being proportional to $\sin(\sigma t) \sin(\sigma t + \alpha)$. The reason for this expression, just as that used earlier for a short channel, that is, $\sin(\sigma t) \cos(\sigma t + \alpha)$, is to have a small value for $\alpha$ for the sake of discussion. For an infinite tidal channel without friction, tide is purely progressive and $\alpha$ in $\sin(\sigma t) \sin(\sigma t + \alpha)$ is identically zero. When the channel is long, the bottom friction damps the reflected wave significantly and the wave at the open end is close to a pure progressive wave—that is, $\alpha$ is close to zero. This ensures that $\overline{\sin(\sigma t) \sin(\sigma t + \alpha)} \sim \overline{\sin^2(\sigma t)} = 1/2 > 0$, and thus $u_{T1} > 0$ across the channel (Fig. 6a; ocean side of $x = 50$ km). As the position goes farther inside the channel, the reflected wave becomes more important relative to the incident wave and $u_{T1}$ is still positive but its magnitude decreases. Farther inside the channel where the wave is close to a standing wave, the situation is similar to a short channel (Fig. 5a), where the positive $u_{T1}$ is in the deep water and the negative $u_{T1}$ is in the shallow water (Fig. 6a).
Similar to short channels, the second term \( u_{T2} \) is all positive and thus the advection induces inward transport of water mass (Fig. 6b). The third term \( u_{T3} \) is again all negative (Fig. 6c), which is consistent with the positive mean surface slope (Fig. 4) into the channel as discussed earlier. Now the magnitudes of \( u_{T2} \) and \( u_{T3} \) are not similar in pattern any more. A larger outward transport by \( u_{T3} \) is through the deeper water, because of an almost laterally independent along-channel pressure gradient. It turns out that, for long channels, the first term \( u_{T1} \) has larger values at shallow waters near the open end (Fig. 6a). This is apparently caused by a larger nonlinear effect due to finite tidal amplitude in shallow water than is present in deep water of a progressive wave. The second term, or the advective transport, \( u_{T2} \) is relatively small relative to either the first or the third term. This is in contrast to the short-channel cases. The third term has a negative value with a larger magnitude than that of \( u_{T1} \) in the deep water. The net effect at the open end is thus an outward flow in the deep water and an inward flow over the shoals (Fig. 6d). As the position goes farther into the channel, the progressive nature of the tide decreases until at a point where the flow resembles that of a short channel again: with an inward flow in the deep water and an outward flow in the shallow water.

c. Comparison with previous studies

The difference between the earlier work (Li and O’Donnell 1997) and this work is that the former is overly simplified such that the effect of the depth dependence of the first-order tidal velocity is neglected, and that results in an underestimate (not an exclusion) of the advective effect, particularly in a short channel in which the advection is relatively more important than that in a long channel. The earlier model also shows dependence of the residual flow on the length ratio, but the exchange flow does not reverse direction between short and long channels. The depth-averaged along-channel component of the residual transport velocity in the earlier model is as follows:

\[
\begin{align*}
u_T &= 2 \frac{\zeta \alpha p}{h} - \frac{h}{\beta} \left( u_p \frac{\partial u_p}{\partial x} + g \frac{\partial \zeta}{\partial x} \right),
\end{align*}
\]

in which \( \zeta \) and \( u_p \) are the “prediction” solution for tide or the approximate solution of the first-order equation that ignores cross-channel depth variation and \( u^2 \) and \( \zeta^2 \) are the second-order residual velocity and water elevation, respectively. The differences between the present result—for example, the first equation in (18)—and the earlier result in (29) include the facts that 1) the earlier results do not have the advective force term in the cross-channel direction \( [u_p(\partial u_p/\partial y)] \) and 2) the earlier model has a \( y \)-independent \( \zeta u^2 \) term. Just as in the present model, all the three terms \( (u_{T1}, u_{T2}, \text{and } u_{T3}) \) contribute to the residual circulation. The force terms in the momentum equations are force per unit mass. The total force in the water column is integrated over the water column, which has less mass in shallower water than in deeper water. The total force in the water column is thus depth dependent. The residual circulation is generated because of this nonuniform depth distribution. This is made clearer when the vorticity is examined as in the earlier work: all three terms contribute to the production of residual vorticity or circulation. The current work is not fundamentally different in that aspect, although we have demonstrated new results when the model is refined to solve the first-order equations without the approximation employed in the earlier study.

The new result of this model is that the flow regimes can change along a long channel or they can be different between short and long channels at the open end. This explains why the residual flow patterns in the James River estuary (Li et al. 1998) are different from that of a short channel (Winant and Gutiérrez de Vasco 2003). It also demonstrates that the tidally induced flow can reinforce the density-driven flow (inward flow through the deep water but outward flow through the shallow water; Hamrick 1979; Wong 1994) in a short channel or at the closed end of a long channel while competing with the density-driven flow at the open end of a long channel. This may be used to explain the flow regime change in Winny Bay (Kim and Voulgaris 2005) between the upper bay and lower bay.

Note also that Zimmerman (1981) has predicted a residual flow pattern consistent with that of a short channel using vorticity dynamics in a flat-bottom semiclosed basin with only lateral friction. In that model, the only nonlinearity considered is the advection. By neglecting other nonlinear terms, that model effectively excludes long channels. The physical explanation of the residual flow pattern of Zimmerman’s model is as follows: the current in the center of the channel is larger than that close to the lateral boundary because of less side-wall friction away from the boundary. This produces a vorticity in opposite senses on the two sides of the axis of the channel, which also alternates the sense of rotation between flood and ebb tides. Because the water particles advecting through the basin affected by the lateral friction will have reduced velocity when it returns to the ocean during ebb tide, the net effect will be an inward flow in the center of the channel and a return flow near the side walls across the
channel. This explanation holds if one substitutes the lateral friction with bottom friction and considers a depth function that varies across the channel, in which case the net inward flow will be through the deep water (not necessarily in the center of the channel) and outward flow will be through the shallow water. This can be seen from Fig. 5b, which shows that the net advective transport is larger in the deep water than in the shallow water, although here the advective term alone does not meet the mass-balance requirement.

d. Multiple-frequency problems

The solution given here only contains a single tidal frequency and the calculations have only been applied to a semidiurnal tidal frequency. In many coastal waters, semidiurnal tide is indeed the main constituent. In other coastal waters, diurnal tide is dominant or tide is a mixture of different constituents of comparable magnitudes. For these situations, the present results do not apply. The solution, however, can be readily modified for these problems because the solution for the second-order equations is not done in the frequency domain. It is an explicit form that only depends on the first-order tide, regardless of whether or not it is a solution of a single frequency. Because the first-order tide is solved from a set of linear equations in frequency domain, we can add all the major tidal constituents of the first-order tide for different frequencies together before solving the second-order subtidal flow field. However, for a multiple-frequency problem in which different frequencies may have comparable magnitudes in the energy spectrum, the definition of residual flow can be problematic because there is no obvious choice for the time period over which the average can be made to yield a “mean” quantity.

e. Other parameters

In this study, we have focused on the effect of a single parameter, \( \delta = 4L/\lambda \). We now take a closer look at the solution for other parameters that may also affect the solution. The first-order tide, (12), can be expressed in a nondimensional form. First notice that \( \omega L \) in (12) can be written as

\[
\omega L = L \sqrt{\frac{i \sigma D}{\gamma}} = L \sqrt{\frac{i \sigma D}{gh} \int_0^D \frac{gh}{ir + \beta/h} \, dy}.
\]

Because both the real and imaginary parts of the function being integrated in the above equation [that is, \( gh/(ir + (\beta/h)) \)] are continuous and can be integrated as a function of \( h(y) \), we can apply a theorem of the mathematics that the integral can be expressed as the function at an intermediate point \( [h = h_0(y_0)] \) multiplied by the interval; that is,

\[
\omega L = L \sqrt{\frac{i \sigma D}{\gamma}} = L \left( \frac{i \sigma D}{gh_0} \right) \int_0^D \frac{gh}{ir + \beta/h} \, dy
\]

\[
= 2\pi L \frac{1}{\lambda} \left( 1 - \frac{i \beta}{\alpha h_0} \right).
\]

where \( h_0 \) is a depth value between the minimum and maximum depths and \( \lambda = \sqrt{gh_0/\beta} \) is the wavelength. The reason that the above equation is approximate is that the function being integrated is a complex one and the intermediate points for the real and imaginary parts of the integration can be different.

Using the above result, (12) can be expressed as

\[
\frac{A}{h_0} = \frac{\xi_0}{h_0} \cos \left[ 2\pi \frac{L}{\lambda} \left( 1 - \frac{i \beta}{\alpha h_0} \frac{x}{L} \right) \right].
\]

The intermediate depth \( h_0 \) is roughly equal to the mean depth. Our calculations show that it is a very good approximation. This equation indicates several nondimensional parameters: 1) the nonlinear parameter \( \varepsilon = \xi_0/h_0 \), 2) the length ratio \( \delta/\lambda = L/\lambda \), and 3) the time-scale ratio \( \tau = \beta/\alpha h_0 \) (the ratio between the tidal time scale \( 1/\sigma \) and the frictional decay time scale \( h_0/\beta \)). All of these parameters are also present in the earlier model of Li and O’Donnell (1997). In fact, the right-hand side of (32) is the same as the solution of Li and O’Donnell (1997). It can be shown that a similar operation to the first-order tidal velocity does not indicate any new parameter except a velocity scale that is a combination of the above parameters. The second-order tidally induced flow is dependent on the first-order tide, and therefore all of the above parameters affect the second-order solution.

In our presentation, we have only used results from a standard bottom drag coefficient \( C_D = 0.0025 \). By varying depth function (and \( h_0 \)) and \( C_D \) (equivalently varying the third parameter: the time-scale ratio \( \beta/\alpha h_0 \)), the length scales will be modified (e.g., the length scale that divides between short and long channels). For given bottom drag coefficient \( C_D \) and velocity magnitude \( (U_0) \) or friction coefficient \( \beta \), the time-scale ratio \( \tau \) is inversely proportional to mean water depth \( h_0 \). For the \( \beta \) value used in this work (0.001 76 m s \(^{-1}\)), \( \tau \) varies from 2.42 to 0.6 when the mean depth changes from 5 to 20 m. In particular, when the mean depth is 8.1 m as is the
5. Conclusions

We have solved an analytic model that allows the use of arbitrary cross-channel depth functions to resolve the subtidal flow regimes in well-mixed tidally dominated channels of different lengths. The model is validated by verifications of momentum balance, continuity equation, and the overall mass balance across the channel. Different depth functions are used to represent different shapes of the cross-channel bathymetry. The solutions are relatively insensitive to the choice of depth profile, and therefore we have only discussed the results of a selected depth function in detail. By varying the channel length we are able to discuss the responses of the residual flows and their contributing components.

The results show that the tidally induced flows in elongated channels are strongly dependent on the length for a given value of friction. The flow direction reverses between short and long channels as well as from the open end to the head in a long channel. The length of the channel is measured by the ratio between the geometric length of the channel and a quarter of the tidal wavelength. When this ratio is smaller than 0.6–0.7, the channel is considered to be short, in which case the tide is close to a standing wave. The short channel exhibits an outward flow over the shallow water and inward flow through the deep water. When this ratio is larger than 0.6–0.7, the channel is considered to be long, the tide is close to a progressive wave, and the tidally induced flow is dominated by the nonlinear wave propagation and the mean pressure gradient at the open end. At the open end of a long channel, an inward flow occurs in shallow water and outward flow occurs in deep water, just the opposite of that of a short channel.

The interior of the long channel close to the head, however, has a flow pattern consistent with that of a short channel. When the channel is intermediate in length, or \( \delta \) is about 0.6–0.7, the residual flow at the open end reaches its minimum, causing a maximum water residence time. When a channel’s depth distribution is altered (such as by a dredging activity or a storm event that involves a large scale transport of sediment), the length ratio may change even if the geometric length of the channel remains the same. The exclusion of density-gradient-related mechanism allows a thorough analysis of the response of tidally induced flows to different channel lengths. As a result, this work only applies to well-mixed tidal channels. It may, however, provide useful reference and methods to study those more complicated flow conditions, such as when the density gradient is included.

Acknowledgments. This study was supported for the most part by the University of Connecticut through several fellowships to Chunyan Li. It was also supported by the Center for Coastal Physical Oceanography at Old Dominion University, Georgia Sea Grant (NA06RG0029-RR746-007/7512067), South Carolina Sea Grant (NOAA NA960PO113), and Georgia DNR (RR100-279-9262764). Constructive comments and suggestions from two reviewers are highly appreciated. George Li provided some useful editorial suggestions.

APPENDIX

The Perturbation Equations

This appendix shows derivation from (5) to the first-order and second-order (8) and (16). It is done using nondimensional variables as a formal procedure of a normal perturbation method. The variables are made dimensionless using the following transformations:

\[
\dot{i} = \sigma, \quad \dot{x} = \frac{x}{L_0}, \quad \dot{y} = \frac{y}{D}, \quad \dot{u} = \frac{u}{U_0}, \quad \dot{v} = \frac{v}{V_0},
\]

\[
\dot{h} = \frac{h}{h_0}, \quad \dot{\xi} = \frac{\xi}{\xi_0}, \quad \dot{U}_0 = \frac{\xi_0}{\xi_0} \sqrt{\frac{g}{h_0}}, \quad \dot{V}_0 = \epsilon \sigma D,
\]

\[
e = \frac{\xi_0}{h_0}, \quad L_0 = \frac{\lambda}{2\pi}, \quad T = \frac{T}{2\pi}, \quad \text{and} \quad T = \frac{2\pi}{\sigma}, \quad (A1)
\]

where \( \sigma, L_0, D, U_0, V_0, \xi_0, \) and \( T \) are the angular frequency of semidiurnal tide, tidal length scale, width of the estuary, scale of the longitudinal velocity, scale of the lateral velocity, tidal amplitude at the mouth, and the period of tide. The choice of the scales is the same
as in Ianniello (1977), which are based on the scales of semidiurnal tide and the geometry of the channel.

Equations (5) can then be written as

\[ \frac{\partial \hat{u}}{\partial t} + \frac{U_0}{\sigma L_0} \frac{\partial \hat{u}}{\partial x} + \frac{V_0}{\sigma D} \frac{\partial \hat{u}}{\partial y} = - \frac{g \zeta_0}{\sigma U_0 L_0} \frac{\partial \hat{\zeta}}{\partial x} + \frac{\beta}{\sigma h_0} \hat{u} + \frac{\beta \zeta_0}{\sigma h_0^2} \hat{\dot{\zeta}}. \]

\[ \frac{\partial \hat{v}}{\partial t} + \frac{U_0}{\sigma L_0} \frac{\partial \hat{v}}{\partial x} + \frac{V_0}{\sigma D} \frac{\partial \hat{v}}{\partial y} = - \frac{g \zeta_0}{\sigma V_0 D} \frac{\partial \hat{\zeta}}{\partial y} - \frac{\beta}{\sigma h_0} \hat{v} + \frac{\beta \zeta_0}{\sigma h_0^2} \hat{\dot{\zeta}}. \]

\[ \frac{\partial \hat{\zeta}}{\partial t} + \frac{h_0 U_0}{\sigma \zeta_0 L_0} \frac{\partial (\hat{h} + \hat{\zeta})}{\partial x} + \frac{h_0 V_0}{\sigma \zeta_0 D} \frac{\partial (\hat{h} + \hat{\zeta})}{\partial y} = 0. \]  

(A2)

It is well established that (Ianniello 1977)

\[ \frac{U_0}{\sigma L_0} = \varepsilon, \quad \frac{V_0}{\sigma D} = \varepsilon, \quad \frac{g \zeta_0}{\sigma U_0 L_0} = 1, \quad \frac{h_0 U_0}{\sigma \zeta_0 L_0} = 1, \quad \text{and} \quad \frac{h_0 V_0}{\sigma \zeta_0 D} = 1. \]  

(A3)

To reduce the complexity of the notation, we now write the coefficient of bottom friction using

\[ \frac{\beta}{\sigma h_0} = \mathcal{D} \]  

(A4)

and the coefficient of the across-channel pressure gradient term using

\[ \frac{g \zeta_0}{\sigma V_0 D} = \mathcal{H} = \left( \frac{\sqrt{g \zeta_0}}{\sigma D} \right)^2. \]  

(A5)

Equation (A2) is therefore equivalent to

\[ \frac{\partial \hat{u}}{\partial t} + \varepsilon \frac{\partial \hat{u}}{\partial x} + \varepsilon \frac{\partial \hat{u}}{\partial y} = - \frac{\partial \hat{\zeta}}{\partial x} + \varepsilon \frac{\partial \hat{\dot{\zeta}}}{\partial x^2}, \]

\[ \frac{\partial \hat{v}}{\partial t} + \varepsilon \frac{\partial \hat{v}}{\partial x} + \varepsilon \frac{\partial \hat{v}}{\partial y} = - \frac{\partial \hat{\zeta}}{\partial y} + \varepsilon \frac{\partial \hat{\dot{\zeta}}}{\partial y^2}, \]

and

\[ \frac{\partial \hat{\zeta}}{\partial t} + \frac{\partial \hat{h} + \hat{\zeta}}{\partial x} + \frac{\partial \hat{h} + \hat{\zeta}}{\partial y} = 0. \]  

(A6)

Because we are interested in the nonlinear effect due to a finite tide–depth ratio, the assumption is that the parameter \( \varepsilon \) is not negligible but is smaller than 1:

\[ \varepsilon = \frac{\zeta_0}{h_0} < 1, \]  

(A7)

in which \( \zeta_0 \) and \( h_0 \) are the tidal amplitude and mean water depth, respectively. In a typical tidal channel, \( \varepsilon \) is on the order of 0.1. We now define a power series expansion for the velocity and elevation in terms of \( \varepsilon \) as

\[ \hat{u} = \hat{u}_1 + \varepsilon \hat{u}_2 + \varepsilon^2 \hat{u}_3 + \cdots, \]

\[ \hat{v} = \hat{v}_1 + \varepsilon \hat{v}_2 + \varepsilon^2 \hat{v}_3 + \cdots, \]

and

\[ \hat{\zeta} = \hat{\zeta}_1 + \varepsilon \hat{\zeta}_2 + \varepsilon^2 \hat{\zeta}_3 + \cdots. \]  

(A8)

in which

\[ \hat{u}_1 = \hat{u}_1(\hat{x}, \hat{y}, \hat{t}), \]

\[ \hat{v}_1 = \hat{v}_1(\hat{x}, \hat{y}, \hat{t}), \]

and

\[ \hat{\zeta}_1 = \hat{\zeta}_1(\hat{x}, \hat{y}, \hat{t}). \]  

(A9)

where \( i = 1, 2, 3, \ldots \) By substituting (A8) into (A6), the \( O(\varepsilon^i) \) and \( O(\varepsilon^3) \) equations are obtained as

\[ \frac{\partial \hat{u}_1}{\partial t} = - \frac{\partial \hat{\zeta}_1}{\partial x} - \mathcal{D} \frac{\partial \hat{u}_1}{\partial \hat{h}}, \]

\[ \frac{\partial \hat{v}_1}{\partial t} = - \frac{\partial \hat{\zeta}_1}{\partial y} - \mathcal{D} \frac{\partial \hat{v}_1}{\partial \hat{h}}, \]

and

\[ \frac{\partial \hat{\zeta}_1}{\partial t} + \hat{h} \frac{\partial \hat{u}_1}{\partial \hat{x}} + \hat{h} \frac{\partial \hat{v}_1}{\partial \hat{y}} = 0. \]  

(A10)

and

\[ \frac{\partial \hat{u}_2}{\partial t} + \hat{u}_1 \frac{\partial \hat{u}_1}{\partial \hat{x}} + \hat{v}_1 \frac{\partial \hat{u}_1}{\partial \hat{y}} = - \frac{\partial \hat{\zeta}_2}{\partial x} - \mathcal{D} \frac{\partial \hat{u}_2}{\partial \hat{h}} + \mathcal{D} \frac{\partial \hat{u}_1 \hat{\zeta}_1}{\partial \hat{h}}, \]

\[ \frac{\partial \hat{v}_2}{\partial t} + \hat{u}_1 \frac{\partial \hat{v}_1}{\partial \hat{x}} + \hat{v}_1 \frac{\partial \hat{v}_1}{\partial \hat{y}} = - \frac{\partial \hat{\zeta}_2}{\partial y} - \mathcal{D} \frac{\partial \hat{v}_2}{\partial \hat{h}} + \mathcal{D} \frac{\partial \hat{v}_1 \hat{\zeta}_1}{\partial \hat{h}}, \]

and

\[ \frac{\partial \hat{\zeta}_2}{\partial t} + \hat{h} \frac{\partial \hat{u}_2}{\partial \hat{x}} + \hat{h} \frac{\partial \hat{v}_2}{\partial \hat{y}} + \hat{h} \frac{\partial \hat{\zeta}_1 \hat{\zeta}_1}{\partial \hat{h}} = 0. \]  

(A11)

If one applies a temporal average over a tidal cycle, (A11) leads to

\[ \frac{\partial \hat{u}_1}{\partial \hat{x}} + \hat{u}_1 \frac{\partial \hat{u}_1}{\partial \hat{x}} = - \frac{\partial \hat{\zeta}_2}{\partial x} - \mathcal{D} \frac{\partial \hat{u}_2}{\partial \hat{h}} + \mathcal{D} \frac{\partial \hat{u}_1 \hat{\zeta}_1}{\partial \hat{h}}, \]

\[ \frac{\partial \hat{v}_1}{\partial \hat{x}} + \hat{v}_1 \frac{\partial \hat{v}_1}{\partial \hat{x}} = - \frac{\partial \hat{\zeta}_2}{\partial y} - \mathcal{D} \frac{\partial \hat{v}_2}{\partial \hat{h}} + \mathcal{D} \frac{\partial \hat{v}_1 \hat{\zeta}_1}{\partial \hat{h}}, \]

and

\[ \frac{\partial \hat{\zeta}_1}{\partial \hat{x}} + \hat{h} \frac{\partial \hat{u}_2}{\partial \hat{x}} + \hat{h} \frac{\partial \hat{v}_2}{\partial \hat{y}} + \hat{h} \frac{\partial \hat{\zeta}_1 \hat{\zeta}_1}{\partial \hat{h}} = 0. \]  

(A12)

Equations (A10) and (A12) are the nondimensional forms of the dimensional equations in (8) and (16),
respectively, which can be verified by changing the variables back to dimensional.

Referring to (3) and (4), the boundary conditions for the nondimensional equations are

\[ \hat{u}_{|\xi=1} = 0, \quad \hat{u}_{|\phi=0,1} = 0, \quad \hat{\xi}_{|\xi=0} = \text{Re}(e^{i\phi}), \quad \text{and} \quad (A13) \]

\[ \int_0^1 \left( \frac{1}{h} + \hat{\xi} \hat{u} \right) d\phi = 0. \quad (A14) \]

The first-order boundary conditions are

\[ \hat{u}_{|\xi=1} = 0, \quad \hat{u}_{|\phi=0,1} = 0, \quad \hat{\xi}_{|\xi=0} = \text{Re}(e^{i\phi}), \quad \text{and} \quad (A15) \]

\[ \int_0^1 \hat{h} \hat{u}_1 d\phi = 0. \quad (A16) \]

The last condition (A16) is automatically satisfied because the first-order tide is oscillatory (a sine function) and its temporal average is always zero; that is, \( \overline{\hat{u}_1} = 0 \). The second-order boundary conditions are

\[ \hat{u}_{|\xi=1} = 0, \quad \hat{u}_{|\phi=0,1} = 0, \quad \hat{\xi}_{|\xi=0} = 0, \quad \text{and} \quad (A17) \]

\[ \int_0^1 (\hat{h} \hat{u}_2 + \hat{\xi} \hat{u}_1) d\phi = 0. \quad (A18) \]

REFERENCES


Hamrick, J. M., 1979: Salinity intrusion and gravitational circula-


