Lecture 12. Equations of Motion

Frictionless Motion → Inertial and Geostrophic Flow

We will now study specific types of flow in the ocean. We are able to classify them according to the relevant forces that cause the motion. To define a type of motion, a series of assumptions have to be made. First we will study frictionless motion, i.e., the motion that is supposed to be unaffected by frictional effects. There are two types of frictionless flow that we will study: inertial flow and geostrophic flow. Inertial flow is the motion that is supposed to be exclusively accelerated by the Coriolis force per unit mass. The geostrophic flow is caused by a balance between the pressure gradient force per unit mass and the Coriolis acceleration. Let us look at each type of motion separately.

Inertial Flow

Inertial flow develops after the relaxation of a forcing agent (e.g. wind) that produces a motion. The only acceleration that modifies this motion is the Coriolis acceleration if it is assumed that friction does not act immediately on the flow. To explore this concept a bit further, imagine the wind blowing on the surface of the ocean. This wind produces currents in approximately the same direction of the wind and with a speed that is roughly proportional to that of the wind. When the wind stops, if we neglect friction and assume homogeneous fluid, the wind-induced current will be modified by Coriolis acceleration and its trajectory will describe a circumference. Hence the assumptions involved in the inertial motion are frictionless flow (very small $E_r$ and $E_	heta$), homogeneous fluid with no sea surface slopes (negligible pressure gradients), no vertical motion. Then, with these assumptions, the equations of motion that describe inertial motion reduce to:

\[
\frac{du}{dt} = f v \\
\frac{dv}{dt} = -f u
\]

which describe accelerations being modified by Coriolis force (per unit mass) only. The solution of the system of two differential equations with two variables (12.1) is:

\[
u = \bar{U} \sin(\bar{U}t); \quad \nu = \bar{U} \cos(\bar{U}t),
\]

where $\bar{U} = (u^2 + v^2)^{\frac{1}{2}}$ is the flow speed. This means that the motion is described by a circle of radius $\bar{U}$, i.e., $u^2 + v^2 = \bar{U}^2$. The motion is clockwise in the northern hemisphere at the constant speed $\bar{U}$. The trajectory of inertial motions maintains its circular shape of radius $\bar{U}$ due to the centripetal acceleration $\bar{U}^2/r$, which is provided by the Coriolis acceleration, i.e.,
\[
\frac{\bar{U}^2}{r} = f\bar{U}.
\]

The inertial radius is then \(r = \bar{U}/f\) and the inertial flow speed is \(\bar{U} = rf\). The inertial period \(T_i\) equals the distance travelled by the inertial circle or \(2\pi r\) divided by the speed of the inertial flow \(rf\), which yields
\[
T_i = \frac{2\pi}{f} = \frac{2\pi}{2\Omega \sin \lambda} = \frac{\pi}{\frac{2\pi}{24 \text{ hrs}} \sin \lambda} = 12 \text{ hrs}.
\]

The inertial period at the poles is minimum (12 hrs) because the Coriolis acceleration is maximum. The period \(T_i\) increases towards the equator. It equals 24 hrs at a latitude of 30° and is 137 hrs at 5°. In our latitude (37°), the inertial period is roughly 20 hrs. If somehow we can determine the radius of the inertial circle, say from Lagrangian measurements, then we can infer the inertial speed from
\[
\bar{U} = \frac{2\pi r}{T}.
\]

Inertial oscillations may also be identified from a current meter record by plotting progressive vector diagrams (PVD), which consist of the cumulative vectorial addition of consecutive measurements.

**Geostrophic Flow**

The geostrophic flow arises as a response to pressure gradients in the ocean. The flow that moves down a pressure gradient and is deflected by the Coriolis acceleration is said to be in **geostrophic balance**. The momentum balance that describes geostrophic flows is:

\[
-fv = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \quad fu = -\frac{1}{\rho} \frac{\partial p}{\partial y}.
\]  

(12.2)

Note that geostrophic flow requires the following assumptions: **steady state** neglects local accelerations; **linear motion** (\(Ro \ll 1\)) ignores flow shears (adveective terms); **flow with negligible**
friction; and hydrostatic balance in the vertical. These assumptions are frequently used to describe flows in regions relatively unaffected (on average) by wind forcing or bottom friction. Therefore, geostrophic flow is a steady flow maintained by the pressure gradient force (per unit mass) acting in one direction and the Coriolis acceleration acting in the opposite direction. The balance of forces (per unit mass) acting on geostrophic flows is depicted in the following diagram

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Pressure Gradient ←⊗→ Coriolis
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This diagram represents accelerations acting with the same magnitude in opposite directions, i.e., a flow with no accelerations (steady) moving into the page.

The geostrophic flow depicts clockwise (or anticyclonic) motion in the northern hemisphere around a center of high pressure. Analogously, geostrophic motion is counterclockwise around a high pressure center in the southern hemisphere (still anticyclonic). Conversely, geostrophic motion around a low pressure system in the northern hemisphere is cyclonic or counterclockwise, i.e.,

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in the northern hemisphere. In the southern hemisphere, the arrows point in the opposite direction. Notice that the flow (in geostrophic balance) is perpendicular to the pressure gradient. Then, if we know the direction and magnitude of the pressure gradient (∂p/∂n, where ∂n is the direction along the pressure gradient), then we can determine the geostrophic flow \( U_h \), which is a horizontal flow, from

\[
fU_h = -\frac{1}{\rho} \frac{\partial p}{\partial n}.
\]

(12.3)

From (12.2) and (12.3) we can see that the key to determine geostrophic flows is to estimate the horizontal pressure gradients. If we can directly measure pressure at two locations, for example, then our problem of determining horizontal pressure gradients and geostrophic flow is solved. This procedure will yield, however, vertically integrated horizontal pressure gradients and one estimate of the vertically integrated flow (throughout the water column), disregarding
its vertical structure. It is important to determine the vertical structure of the flows because we may encounter near-surface flow in one direction and interior flow travelling in opposite direction. This situation would produce near-zero vertically integrated flows that do not reflect the actual flow field.

The difficulty of elucidating the vertical structure of geostrophic flows is overcome with the help of the hydrostatic approximation, which relates pressure fields to density fields. Then, all we need to measure is density profiles at different locations to determine horizontal pressure gradients among locations. We will first study the simple case of determining vertically integrated horizontal pressure gradients (barotropic pressure gradients) and later on we will learn how to evaluate depth-dependent horizontal pressure gradients (baroclinic pressure gradients).

**Barotropic pressure gradients in geostrophic flow calculation**

To understand this concept, we have to present the following definitions. Geopotential surface is a surface on which the acceleration due to gravity is perpendicular everywhere. It is a level surface where no motion will occur. We will denote a geopotential surface with the greek letter \( \Phi \). An isobaric surface is a surface on which the pressure is the same everywhere. In the ocean, *barotropic conditions* are those in which isobaric surfaces and isopycnal surfaces are parallel and they form an angle with respect to geopotential surfaces. If a geopotential surface is parallel to an isobaric surface, then there is no motion.

![Diagram of geopotential surface and barotropic conditions](image)

To determine barotropic pressure gradients, assume a situation in which we can measure the bottom pressure, or the sea level at two locations (1 and 2), and that the sea level is not parallel to a geopotential:
$P_2 - P_1 = \Delta p = \rho g \Delta z$, and then

$$-\frac{1}{\rho} \frac{\Delta p}{\Delta x} = -g \frac{\Delta z}{\Delta x} = -g \tan \gamma.$$ 

It has to be assumed that points 1 and 2 are on an isobaric surface at a depth $z$, and that the isobaric surface is parallel to a geopotential surface. Because of isobaric and geopotential surfaces being parallel at one level, there is no fluid motion along that level. This level is called the level of no motion. Bear in mind that geostrophic currents are relative to a level of no motion that is sufficiently deep (at least 200 m). The geostrophic balance can then be expressed (from 12.3) as:

$$fU_h = -\frac{1}{\rho} \frac{\partial p}{\partial n} = -g \frac{\Delta z}{\Delta n} = -g \tan \gamma$$

so that in order to calculate $U_h$, we need to determine the slope of the isobars, that is, $\gamma$ or $\Delta z/\Delta n$. In practice, it is difficult to determine the slope. A technique to determine the absolute slope of the sea surface is to use laser altimetry from satellites (Figure in home page). Regions of relatively higher elevation (high dynamic topography) represent anticyclonic circulation. Note that flow is parallel to isobars and that flow speed is inversely proportional to the distance between isobars (also seen in 12.3), i.e., flow is strongest where horizontal pressure gradients are greatest. This representation is similar to isobaric maps used by meteorologists and from which wind velocities can be inferred. In fact, the geostrophic equation can be applied with less difficulty in meteorology because the air pressure can be measured directly at different elevations. These measurements are used to calculate horizontal pressure gradients. Meteorologists approximate mean sea level as a geopotential surface because ocean currents produced by sea level differences are much weaker than wind velocities. An important point to make about satellite measurements of sea level is that they allow estimates of geostrophic currents only at the surface. For example in the Gulf Stream and Kuroshio regions, the sea level slopes in the transverse direction to the flow are approximately one meter in 100 km, which give geostrophic flows of 1 m/s. This is a slight underestimate of the real flows, which reach 1.5 to 2 m/s. As mentioned before, we have to measure density profiles at different locations to determine the baroclinic pressure gradients and the vertical structure of geostrophic flows. This will be discussed next.
Baroclinic pressure gradients in geostrophic flow calculation

Some definitions are also required to explain this concept. \textbf{Geopotential distance} is the distance between two isobaric surfaces located at depths \( z_1 \) and \( z_2 \). The work required to raise a water parcel of mass \( M \) by a distance \( dz \) against gravity is \( Mgdz \). The change of geopotential \( d\Phi \) is the potential energy per unit mass gained by the parcel, or:

\[
d\Phi = g \, dz = -\frac{1}{\rho} \, dp.
\]

The geopotential distance \( d\Phi \) (Joules/kg or \( m^2/s^2 \)) is the value we have to determine between given depths \( (z_1 \) and \( z_2) \) in the water column so that we can calculate the baroclinic pressure gradients between stations. To do that, we have to integrate \( d\Phi \) between \( z_1 \) and \( z_2 \),

\[
\int_{1}^{2} d\Phi = \int_{1}^{2} g \, dz = -\int_{1}^{2} \frac{dp}{\rho},
\]

but remember from the properties of seawater that \( 1/\rho = a = a_{35,0,p} + \delta \), i.e., specific volume of a given parcel of seawater equals the specific volume at standard conditions of salinity, temperature and pressure, plus the specific volume anomaly. The result of the integration in (12.4) is then

\[
\Phi_2 - \Phi_1 = g(z_2 - z_1) = -\int_{1}^{2} a_{35,0,p} \, dp - \int_{1}^{2} \delta \, dp,
\]

which represents the geopotential distance between \( z_1 \) and \( z_2 \) where the pressures are \( p_2 \) and \( p_1 \). The first term after the second equality is the standard geopotential distance \( (\Delta \Phi_{std}) \), and the second term is the geopotential anomaly \( (\Delta \Phi) \). As we will see, \textbf{the calculation of} \( \Delta \Phi \) \textbf{is the key point in the estimate of geostrophic flows}. Notice that geopotential distance is not really a distance because it has units of energy per unit mass, or Joules per kilogram, or \( m^2/s^2 \). For example, for a distance \( dz = 1 \) m, \( \Phi_2 - \Phi_1 = 9.8 \) J/kg \( \approx 1 \) dynamic meter = 10 J/kg = 10 \( m^2/s^2 \). Depth is equivalent to dynamic meters. Therefore, bear in mind that dynamic topography refers to geopotential distances. Regions of relatively high dynamic topography in the ocean are equivalent to high pressure systems in the atmosphere. Fluids will flow from regions of relatively high dynamic topography to regions of low dynamic topography. Note that if isobaric surfaces coincide with geopotential surfaces, then there will be no motion (stationary state). In this case, there is no dynamic topography.

In the case of baroclinic conditions, which is a realistic situation in the ocean, the isobars are oblique to the isopynals. The isobars will then intersect geopotential surfaces at different slopes as a function of depth. This is represented in the following diagram:
The geostrophic equation obtained from the diagram is:

\[
    f(U_1 - U_2) = \frac{1}{\Delta x} \int_{p_1}^{p_2} \delta_2 dp - \int_{p_1}^{p_2} \delta_1 dp = \frac{1}{\Delta x} (\Delta \Phi_2 - \Delta \Phi_1)
\]  

(12.6)

This is the form of the geostrophic equation that is commonly used to calculate geostrophic flows from hydrographic data. Ultimately, we seek to determine the value of the integrals, which represent horizontal pressure gradients. Integrations are carried out numerically with estimates of specific volume anomalies (δ₁, δ₂) obtained from temperature and salinity observations at stations 1 and 2, as in the example seen in class. Remember that the currents estimated with this approach are relative to an assumed level of no motion (LNM) where isobaric surfaces are parallel to geopotential surfaces, and hence no motion there.