Lecture 7. Conservation of Mass (Volume)

To quantify processes in the ocean we usually assume that the volume of fluid we study is conserved. This is a powerful assumption and allows us to get an idea, for example, of approximate transports into and out of a basin simply by knowing the typical salinities inside and outside of the basin along with the net gains and losses of volume due to evaporation/precipitation. The first step towards expressing conservation of volume is to consider a volume element with dimensions \(dx, dy, dz\) fixed in space in a right-handed coordinate system, in which positive \(x, y, z\) are aligned to the east, north, and upwards, respectively (it is sometimes useful to think of this volume element as something that you can see, such as your classroom, where each of its sides is \(dx, dy, dz\)), e.g.,

![Diagram of volume element](image)

The flow in the \(x, y, z\) directions is represented by \(u, v, w\) respectively. The flux of mass that enters the volume element (mass per unit time) through the \(dy \cdot dz\) face of the element is \(\rho \cdot u \cdot dy \cdot dz\). Then, the flux of mass leaving the volume equals the entering flux plus its changes with respect to the \(x\) direction along the distance \(dx\), i.e.,

\[
\rho u dydz + \frac{\partial}{\partial x} (\rho u) dx dy dz.
\]

This is a first order Taylor expansion of the flux \(\rho u\) along \(dx\). Note that the flux units are mass/time. The net flux of mass through the \(dy \cdot dz\) faces of the volume element equals the flux in minus the flux out, i.e.,

\[
Net \ flux \ in \ x \ direction = -\frac{\partial}{\partial x} (\rho u) dx dy dz.
\]

The net fluxes in the \(y\) and \(z\) directions can be derived similarly to obtain:
\[ \text{Net flux in y direction} = -\frac{\partial}{\partial y} (\rho v) dy dx dz, \]

\[ \text{Net flux in z direction} = -\frac{\partial}{\partial z} (\rho w) dz dx dy. \]

Then, the change of mass per unit time flowing through the volume element can directly be represented by \( \frac{\partial (\text{Mass})}{\partial t} = \frac{\partial \rho}{\partial t} dx dy dz \) and equals

\[ \frac{\partial \rho}{\partial t} dx dy dz = -\frac{\partial}{\partial x} (\rho u) dx dy dz - \frac{\partial}{\partial y} (\rho v) dx dy dz - \frac{\partial}{\partial z} (\rho w) dx dy dz. \]

The change of mass per unit time per unit volume \((dx dy dz)\) is

\[ \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} + \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + \rho \frac{\partial w}{\partial z} = 0. \quad 7.1 \]

Note that the first term represents the local change of mass (per unit time per unit volume) of the fluid, and that the second, third, and fourth terms indicate the changes of mass (per unit time per unit volume) due to advection. Remember that these four terms, together, represent the total changes of mass per unit time, i.e., \( \frac{d \rho}{dt} \) or the total derivative of \( \rho \). Then, equation (7.1) can be re-written as:

\[ \frac{1}{\rho} \frac{d \rho}{dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad 7.2 \]

If the fluid is considered incompressible, which is a valid assumption in the shallow parts of the ocean \((\leq 2000 \text{ m})\), the density of a fluid element does not change in time, i.e., \( \frac{d \rho}{dt} = 0 \). This is the basis of Boussinesq approximation that establishes that density variations in the ocean are negligible except when multiplied by the acceleration due to gravity, as in the baroclinic pressure gradient term of the equations of motion that we will study later. With the Boussinesq approximation, equation (7.2) becomes

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{u} = 0. \quad 7.3 \]
This is the **continuity equation**, which is one of the basic equations in fluid mechanics and in oceanography. It is called the continuity equation because it requires the flux into any closed surface of a fluid element to be equal to the flux out. The symbol $\nabla$ denotes the Del or Nabla operator, which is a vector operator with components $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and describes variations of a property in the 3 directions of the reference frame. The special case of variations in $x$ and $y$ only is represented by $\nabla_h$, i.e., horizontal gradient.

The continuity equation is very useful for determining areas of flow convergence and divergence in the ocean. Flow convergence is denoted by $\nabla_h \vec{u} < 0$, and indicates areas of flow deceleration or flows travelling in opposite directions, such as in some oceanic fronts. On the other hand, flow divergence is denoted by $\nabla_h \vec{u} > 0$, and indicates areas of flow acceleration, such as in upwelling regions as the flow escapes from the coast. Convergences cause downward velocities ($-w$) and divergences cause upward velocities ($+w$).

**Example of use of continuity equation.**

It is difficult to measure vertical velocities in the ocean but they can be inferred with the continuity equation from horizontal velocities that can be measured at different sites. Consider a channel 10 m deep in which the flow is not a function of the $y$ direction ($\partial v/\partial y = 0$), and in which the surface flow at an upstream location is 0.25 m/s and at a location 1 km downstream is 0.30 m/s. Estimate the vertical flow ($w$).

This problem should satisfy $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$.

The two flow measurements indicate that $\frac{\partial u}{\partial x} = \frac{\Delta u}{\Delta x} = \frac{0.05 \text{ m/s}}{1000 \text{ m}} = 5 \times 10^{-5} \text{ s}^{-1}$.

Hence,$\frac{\partial w}{\partial z} = -5 \times 10^{-5} \text{ s}^{-1}$. Upon integration of the last expression over 10 m depth, we obtain:

$$w = 5 \times 10^{-4} \text{ m/s} = 43.2 \text{ m/day}.$$ This is a sizable vertical velocity but higher values may be found in coastal and estuarine fronts, where strong density gradients develop.

The concept of conservation of volume also can be expressed for a semi-enclosed basin as bulk quantities that enter and/or leave the basin, i.e., volume in ($V_{in}$) equals volume out ($V_{out}$). The quantity $V_{in}$ can include the oceanic flow into the basin ($V_{o}$), river discharge ($R$), pluvial precipitation (rain or $P$), and thawing of ice ($\Theta$). The quantity $V_{out}$ can include the basin flow out into the ocean ($V_{b}$), evaporation ($E$), and freezing ($\Phi$), e.g.,
For a given basin, it can be assumed then that

$$V + R + P + \Theta = V + E + \Phi. \quad (7.4)$$

What goes in must go out. Note that expression (7.4) originates from (7.2) by assuming steady state conditions, motion along one dimension (no gradients perpendicular to the flow direction), and flow integrated over a closed surface.