4. Practical Spectral Analysis

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April 20, 2007

Measurements produce sequences of numbers

Measurement purpose: characterize a stochastic process.

Example:
Process: water surface elevation as a function of time
Parameters: mean water level, wave variance (RMS height), power spectrum etc...

Commonly obtainable data are:
- not one realization;
- not a finite segment of one realization;
- but a discrete finite segment of one realization.

Measurements = finite sequence of numbers.
Goal: Apply the theory (e.g. spectral analysis) to sequences.
Definition & properties

N-order sequence is:

1. a set of numbers: \( \{g_k\} \), \( g_k \in \mathbb{C} \), \( k = 0, 1, \ldots, N - 1 \).
2. an \( N \)-dimensional vector: \( \mathbf{g} = (g_1, g_2, \ldots, g_{N-1}) \subseteq \mathbb{C}^N \)
3. a complex function \( g \) of an integer argument \( g(k) \).

Term-by-term operations

<table>
<thead>
<tr>
<th>op</th>
<th>set</th>
<th>function</th>
<th>vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>( {a_k} + {b_k} = {a_k + b_k} )</td>
<td>( a + b )</td>
<td>( a + b )</td>
</tr>
<tr>
<td>\times</td>
<td>( {a_k} {b_k} = {a_k b_k} )</td>
<td>( ab )</td>
<td>?</td>
</tr>
<tr>
<td>\alpha \times</td>
<td>( \alpha {a_k} = {\alpha a_k} )</td>
<td>( \alpha a )</td>
<td>( \alpha a )</td>
</tr>
<tr>
<td>/</td>
<td>( {a_k} / {b_k} = {a_k/b_k} )</td>
<td>( a/b )</td>
<td>?</td>
</tr>
</tbody>
</table>

? means no commonly used corresponding op.
\( \alpha \) is a complex constant

Periodicity and symmetry

As a function, \( g \) is defined for \( k = 0, 1, \ldots, N - 1 \).
Extend \( k \in \mathbb{N} \) by “gluing” together copies of \( g \).
Then

- Extended \( g \) is \( N \)-periodic \( g(k + N) = g(k) \).
- Can define symmetries for \( g \)

\[
\begin{align*}
g(t) &= \pm g(t) \\
g(-k) &= g(N - k) = \pm g(k)
\end{align*}
\]

Using “sequence” notation

\[
g_{-k} = g_{N-k} = \pm g_k
\]
Example: Periodicity

\[ k = 3; \ N = 10; \ k + N = 13; \ k - N = -7. \]

Example: Even/Odd

\[ k = 3; \ N = 10; \ k + N = 13; \ k - N = -7; \ N - k = 7 \]
Problem: Fourier representation for sequences

- Measured data is discrete:
  function $g$ of integer argument, $g(k)$, $k = 0, 1, \ldots, N - 1$.
- Is it possible to build a Fourier representation for $g$?
  (DFT = Discrete Fourier Transform)
- What would be its properties?
- What properties of the continuous version transfer to DFT?
- What new properties would DFT have?

Procedure for building DFT

<table>
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<th>Continuous</th>
<th>Discrete</th>
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<td>time/freq. domain</td>
<td>$g(t)$</td>
<td>$g(n) = g(n)$</td>
</tr>
<tr>
<td>Fourier pair</td>
<td>$G(f) = \hat{g}(f)$</td>
<td>$G_k = G(k) = \hat{g}(k)$</td>
</tr>
<tr>
<td>$g \leftrightarrow \hat{g}$</td>
<td>$G(f) = \int_{-\infty}^{\infty} g(t)e^{-2\pi ift}dt$</td>
<td>$G = D[g]$</td>
</tr>
<tr>
<td></td>
<td>$g(ft) = \int_{-\infty}^{\infty} G(t)e^{2\pi ift}df$</td>
<td>$g = D^{-1}[G]$</td>
</tr>
</tbody>
</table>

Remarks:
- The goal is to define and study DFT operator $D$;
- Grids do not enter DFT process: $D$ just maps sequence $\rightarrow$ sequence;
- Grids important for physical interpretation;
- Will prefer $G(k)$ instead of $\hat{g}(k)$;
Step 1: Discretize FT

Discretize Fourier Integrals:

1. **Direct**: Time grid: \( t_n = n\Delta t, \ n = 0, 1, \ldots, N_t - 1; \)

\[
G(f) = \int_{-\infty}^{\infty} g(t) e^{-2\pi i ft} dt \approx \sum_{n=0}^{N_t-1} g(t_n) e^{-2\pi i f n \Delta t}
\]

Time grid defined. Frequency grid NOT defined.

2. **Inverse**: Freq. grid: \( f_k = k\Delta f, \ k = 0, 1, \ldots, N_f - 1; \)

\[
g(t) = \int_{-\infty}^{\infty} G(f) e^{2\pi i ft} df \approx \sum_{k=0}^{N_f-1} G(f_k) e^{2\pi i f k \Delta f}
\]

Frequency grid defined. Time grid NOT defined.

No visible relation between time-freq. grids.

Grids important for physical interpretation.

Step 2: Define grids

Time/freq grids have to satisfy

- **Invertibility**: \( N_t = N_f = N \). DFT = linear transformation of vectors.

- **Physical constraints**:

1. Time grid: defined by measurement.

\[
N \text{ points } t_n = n\Delta t, \ n = 0, 1, \ldots, N - 1; \ T = (N - 1) \Delta t;
\]

2. Frequency grid defined by physics:

   - largest period = \( T \Rightarrow \Delta f = \frac{1}{T}; \)
   - highest frequency on grid: \( f_{\text{max}} = (N - 1)\Delta f = \frac{N - 1}{T}; \)

\[
N \text{ points; } f_k = k\Delta f, \ k = 0, 1, \ldots, N - 1; \ \Delta f = 1/T;
\]
Step 3: Put it all together

Time grid: \( t_n = n\Delta t, \ n = 0,1,\ldots,N-1; \ T = (N-1)\Delta t; \)
Frequency grid: \( f_k = k\Delta f, \ k = 0,1,\ldots,N_f; \ \Delta f = 1/T; \)
\[ f_k t_n = kn\Delta f \Delta t = \frac{kn}{N-1} \]

\[
G(f_k) \simeq \frac{T}{N-1} \sum_{n=0}^{N-1} g(t_n)e^{-2\pi i f_k t_n} = \frac{T}{N-1} \sum_{n=0}^{N-1} g_n e^{-2\pi i \frac{kn}{N-1}}
\]

\[
g(t_n) \simeq \frac{1}{T} \sum_{k=0}^{N-1} G(f_k) e^{2\pi i \frac{kn}{N-1}}
\]

Note:
- in \( G(f_k) \), first & last terms: \( g_0 \) & \( g_{N-1} \) (no exponential);
- If \( g_0 = g_{N-1} \) (periodic sequence), rename \( N-1 = N \).

Recount for a periodic sequence

If the sequence is periodic, we only need \( N-1 \) terms.

Renaming \( N-1 \mapsto N \):
- \( \Delta t = \frac{T}{N-1} \mapsto \Delta t = \frac{T}{N} \)
- \( \sum_{n=0}^{N-1} \mapsto \sum_{n=0}^{N-1} \)
- \( \{g_n\} \) has \( N \) terms
- but \( g_N = g_0 \) (periodic)

Renaming \( N-1 \mapsto N \)
The Discrete Fourier Transform (DFT)

Properties of the DFT

DFT-Specific Properties

Power spectrum estimate

N-order sequences

The DFT-IDFT

Definition and grids for DFT

The DFT form below is from J Weaver (Mathematica).

Definition

1. Grids: \( t_n = n\Delta t, \Delta t = T/N; f_k = k\Delta f; \Delta f = 1/T; \)
2. Grid reciprocity: \( \Delta t \Delta f = 1/N, \) and \( Tf_{\text{max}} = N. \)
3. DFT and IDFT are defined as:
   \[
   G_k = \frac{1}{N} \sum_{n=0}^{N-1} g_n e^{-2\pi i f_k t_n} = \frac{1}{N} \sum_{n=0}^{N-1} g_n e^{-2\pi i \frac{k}{N}} = \frac{1}{N} \sum_{n=0}^{N-1} g_n (\Omega_N^k)^n,
   \]
   \[
   g_n = \sum_{j=0}^{N-1} G_k e^{2\pi i f_k t_n} = \sum_{k=0}^{N-1} G_k e^{2\pi i \frac{k}{N}} = \sum_{k=0}^{N-1} G_k \Omega_N^{kn}.
   \]
4. \( \Omega_N = e^{\frac{2\pi i}{N}} \) is called DFT kernel.

Other DFT forms

See for example Briggs & Henson “The DFT, an owner’s manual”

Mathematica
\[
G_k = \sum_{n=0}^{N-1} g_n e^{-2\pi i \frac{kn}{N}} \quad g_n = \frac{1}{N} \sum_{k=0}^{N-1} G_k e^{2\pi i \frac{kn}{N}}
\]

Matlab
\[
G_k = \sum_{n=1}^{N} g_n e^{-2\pi i \frac{(k-1)(n-1)}{N}} \quad g_n = \frac{1}{N} \sum_{k=1}^{N} G_k e^{2\pi i \frac{(k-1)(n-1)}{N}}
\]

IMSL
\[
G_{k+1} = \sum_{n=0}^{N-1} g_{n+1} e^{-2\pi i \frac{kn}{N}} \quad g_{n+1} = \frac{1}{N} \sum_{k=0}^{N-1} G_{k+1} e^{2\pi i \frac{kn}{N}}
\]

etc...
Matricial form of DFT

1. Use the vector notation \( g = \{g_n\} \) and \( G = \{G_k\} \);
2. introduce the matrix \( \Omega_N = (\Omega_N^{kn})_{k,n=1, \ldots, N-1} \);

\[
\begin{align*}
\text{Sequence} & \quad \text{Matrix} \\
G_k & = \frac{1}{\sqrt{N}} \sum_n g_n (\Omega_N^{kn})^* \\
g_k & = \sum_n G_n (\Omega_N^{kn}) \\
\end{align*}
\]

where \( \Omega_N = \exp(2\pi i / N) \). For example:

\[
\begin{pmatrix}
G_0 \\
G_1 \\
G_2 \\
\vdots \\
G_{N-1}
\end{pmatrix} = \frac{1}{N} \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \Omega_N & \Omega_N^2 & \cdots & \Omega_N^{N-1} \\
1 & \Omega_N^2 & \Omega_N^4 & \cdots & \Omega_N^{2(N-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \Omega_N^{N-1} & \Omega_N^{2(N-1)} & \cdots & \Omega_N^{(N-1)^2}
\end{pmatrix}
\begin{pmatrix}
g_0 \\
g_1 \\
g_2 \\
\vdots \\
g_{N-1}
\end{pmatrix}
\]

DFT Kernel matrix is orthogonal (unitary)

Fourier “basis” functions are orthonormal: \( \int_{-\infty}^{\infty} e^{2\pi i f t} dt = \delta(f) \).

What about the DFT?

Rows \( u_N^k = \{(u_N^k)_n\} = \{\Omega_N^{kn}\} \) of kernel matrix \( \Omega_N \) :

\[
\begin{align*}
u_N^k & = \begin{pmatrix}
e^{2\pi i k \frac{1}{N}} & e^{2\pi i k \frac{2}{N}} & \cdots & e^{2\pi i k \frac{N-1}{N}} \\
e^{2\pi i k t_1} & e^{2\pi i k t_2} & \cdots & e^{2\pi i k t_{N-1}}
\end{pmatrix} \\
\end{align*}
\]

Compare with

\[
U(t_n) = e^{2\pi i f t_n}.
\]

**Theorem**

For any integer \( k \), with \( n \neq 0 \) and \( n \neq N \),

\[
\sum_{n=0}^{N-1} (u_N^k)_n = \sum_{n=0}^{N-1} \Omega_N^{kn} = \sum_{n=0}^{N-1} e^{2\pi i \frac{k n}{N}} = 0.
\]
**Proof.** Form the sequence

\[ V_n = \frac{\Omega_N^{kn}}{\Omega_N^k - 1}, \]

\[ V_{n+1} - V_n = \frac{\Omega_N^{k(n+1)}}{\Omega_N^k - 1} - \frac{\Omega_N^{kn}}{\Omega_N^k - 1} = \frac{\Omega_N^{kn} (\Omega_N^k - 1)}{\Omega_N^k - 1} = \Omega_N^{kn}. \]

\( V_n \) “primitive” of \( \Omega_N^{kn} \); \( V_n \) is periodic, \( V_n = V_{n+N} \).

\[ \sum_{n=0}^{N-1} \Omega_N^{kn} = \sum_{n=0}^{N-1} V_n = V_N - V_0 = 0. \]

---

**DFT as operator**

\[ \{G_k\} = D\{g_n\}; \quad \{g_n\} = D^{-1}\{G_k\} \]

\[ D\{g_n\} = \frac{1}{N} \sum_{n=0}^{N-1} g_n \Omega_N^{kn}. \]

- Notations \( f, g, \{f_j\}, \{g_k\} \) are used for sequences
- The DFT are capitalized \( F, G \)
- A term in the sequence is denoted by the name of the sequence with an index \( f_j, g_k \)

\[ \{G_k\} = \frac{1}{N} \sum_{k=0}^{N-1} g_n \left( \Omega_N^k \right)_n^* \Leftrightarrow G = D[g] \]
DFT: The usual properties (1)

**Linearity:** \[ D[af + bg] = aF + bG. \]

<table>
<thead>
<tr>
<th>FT</th>
<th>DFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F[X(t - a)] = X(f) e^{-2\pi iaf} )</td>
<td>( D{g_{n-m}} = {G_k} \left{ \Omega_N^{-im} \right} )</td>
</tr>
<tr>
<td>( F[X(t)e^{2\pi if_0 t}] = \hat{X}(f - f_0) )</td>
<td>( D{g_n} \left{ \Omega_N^{mn} \right} = {G_{k-m}} )</td>
</tr>
</tbody>
</table>

**Derivative 1:**
\[ \frac{d\hat{X}(f)}{df} = -2\pi iF[tX(t)] \]
\[ D\{G_j\} = D [\{g_k\} \left\{ \Omega_N^{-j} - 1 \right\}] \]

**Derivative 2:**
\[ F\left[ \frac{dX(t)}{dt} \right] = 2\pi if \hat{X}(f) \]
\[ D\{\Delta \{g_k\}\} = \{G_j\} \left\{ \Omega_N^{-1} \right\} \]

For example,

\[ D\{\{f_k\} \{\cos(2\pi kn/N)\}\} = \frac{1}{2} \left[ \{F_{j+n}\} + \{F_{j-n}\} \right] \]

\[ D\{\{f_k\} \{\sin(2\pi kn/N)\}\} = \frac{i}{2} \left[ \{F_{j+n}\} - \{F_{j-n}\} \right] \]

DFT: The usual properties (3)

**Symmetry relations**

<table>
<thead>
<tr>
<th>( g )</th>
<th>( G )</th>
</tr>
</thead>
<tbody>
<tr>
<td>real, even</td>
<td>real, even</td>
</tr>
<tr>
<td>real, odd</td>
<td>imaginary, odd</td>
</tr>
<tr>
<td>imaginary, even</td>
<td>imaginary, even</td>
</tr>
<tr>
<td>imaginary, odd</td>
<td>real, odd</td>
</tr>
</tbody>
</table>
The Discrete Fourier Transform (DFT)

Properties of the DFT

DFT-Specific Properties

Power spectrum estimate

Convolution, Correlation, Parseval Relation

Examples

- Nth-order Delta sequence: with $k = 0, 1, \ldots, N - 1$,

  \[ \{ \delta_k \} = \{ 1, 0, \ldots, 0 \}, \]

- Unit sequence, with $k = 0, 1, \ldots, N - 1; \{ u_k \} = \{ 1, 1, \ldots, 1 \}$,

  Delta and Unit sequences are a DFT pair:

  \[ \frac{1}{N} \{ u_j \} = D \{ \{ \delta_k \} \}. \]

  \[ \{ \delta_j \} = D \{ \{ u_k \} \}. \]

- The DFT of the cosine is the sum of 2 symmetric deltas:

  \[ D \left\{ \cos \left( \frac{2 \pi kn}{N} \right) \right\} = \frac{1}{2} \{ \delta_{j+n} \} + \{ \delta_{j-n} \}. \]
Convolution

Let \( f \) and \( g \) be two \( N \)-order sequences,

**Definition**

\[
(f \circ g)_n = \sum_{m=0}^{N-1} \{f_m\} \{g_{n-m}\}.
\]

**Theorem**

(i) \( D[fg] = D[f] \circ D[g] \),

(ii) \( D[f \circ g] = N D[f] D[g] \),

Correlation

**Definition**

\[
(f \star g)_n = \sum_{m=0}^{N-1} \{f_m\} \{g^*_{n+m}\}.
\]

**Theorem**

(i) \( D[fg^*] = F \star G \),

(ii) \( D[f \star g] = N F G^* \),
**The Discrete Fourier Transform (DFT)**

**Properties of the DFT**

**DFT-Specific Properties**

**Power spectrum estimate**

**Zero padding**

**Aliasing**

---

**Parseval Relation**

**Theorem**

If $f$ and $F$ form a DFT pair of $N$th-order sequences,

$$
\sum_{k=0}^{N-1} |f_n|^2 = N \sum_{k=0}^{N-1} |F_k|^2
$$

---

**DFT-specific properties**

What happens when a sequence is padded with zeros?

Use symmetric DFT

$$
\{G_j\} = D [\{g_k\}];
$$

$$
D[\cdot] = \frac{1}{N + 1} \sum_{k=-N/2}^{N/2} [\cdot] \Omega_N^{-jk}.
$$

- $g_{old}$ $N$-order sequence;
- $g_{new}$ $M$-order sequence zero-padded

$$
g_{new}^{new}_k = \begin{cases} 
0 & -M/2 \leq k \leq -N/2 \\
g_{old}^k & -N/2 \leq k \leq N/2 \\
0 & N/2 \leq k \leq M/2
\end{cases}
$$
Zero padding

If say \( M + 1 = p(N + 1) \), DFT of \( g_{\text{new}} \):

\[
G_{j}^{\text{old}} = pG_{pj}^{\text{new}}, \quad j = -N/2, \ldots, N/2.
\]

Reciprocity relations:

\[
\Delta f^{\text{old}} = p\Delta f^{\text{new}}.
\]

Zero-padding is a technique which refines the frequency grid.

---

a) sequence \( g \); b) its Fourier pair \( G \).

Blue line: “true” shape of \( g \) and \( G \).

Aliasing

Aliasing: misrepresenting an oscillation which cannot be resolved as a different (lower) frequency.

![Aliasing Diagram]

**Definition**

1. The frequency \( f_N = \frac{f_{\text{max}}}{2} = \frac{1}{2\Delta t} \) is called the Nyquist frequency.
2. \( f_N \) is the highest frequency that can be resolved un-aliased by spectral analysis from a sequence sampled at \( \Delta t \).

**Shannon Sampling Theorem**

(Shannon sampling theorem) \( \hat{g}(f) = 0 \) for \( |f| > f_N \). If \( \Delta t \) is chosen so that

\[
\Delta t \leq \frac{1}{f_{\text{max}}},
\]

then \( g(t) \) may be reconstructed exactly from its samples \( g_n = g(n\Delta t) \) by

\[
g(t) = \sum_{n=-\infty}^{\infty} g_n \text{sinc} \left[ \frac{\pi(t - t_n)}{\Delta t} \right].
\]
The Discrete Fourier Transform (DFT)

Properties of the DFT

DFT-Specific Properties

Power spectrum estimate

Step 3: Periodogram

Power spectrum estimate

Continuous

Discrete

- $X(t) = \text{one realization}$;  
- DFT pair $\hat{X} = F[X]$;  
- Sample spectrum $\frac{1}{T} |\hat{X}_T(f)|^2$;  
- $h(f) = E \left\{ \lim_{T \to \infty} \left| \frac{1}{T} \hat{X}_T(f) \right|^2 \right\}$

- $\{g_k\} = \text{one realization}$;  
- DFT pair $\{G_n\} = D[\{g_k\}]$;  
- Periodogram $\approx \{|G_n|^2\}$;  
- Average, let $N \to \infty$.

Parseval, continuous:  

$\int_{-\infty}^{\infty} |X(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{X}(f)|^2 \, df$

Parseval, discrete:  

$\sum_{k=0}^{N-1} |g_n|^2 = N \sum_{k=0}^{N-1} |G_k|^2$

**Definition**

Sample spectrum (periodogram) of sequence $\{g_n\}$ is the sequence $\{P_k\}$:

$$P_k = N \left| G_k \right|^2$$

Periodogram = “natural” estimate of power spectrum:

- $\{P_k\} = \text{DFT of autocorrelation } \{c_n\} = \sum_{n=0}^{N-1} \{g_m\} \{g_{n+m}^*\}$  
- Direct discretization of “continuous” version of definition.
Step 4: Let $N \to \infty \ldots$

\{P_k\} is a consistent estimate of the power spectrum if
\[ \lim_{N \to \infty} \mathbb{E}[P_k] = h(f_k); \text{ and } b) \lim_{N \to \infty} \sigma^2_P = 0 \]

**Example:** White noise process.

Left: $\{P_k\}$ for $N = 20, 50, 100, 500$. Right: variance of $P_k$ versus $N$.

It does not work!

The “raw” periodogram is *extremely poor* as estimate $h(f_k)$:

- $P_k$ is not a consistent estimate: $\sigma^2_P \not\to 0$ as $N \to \infty$.
- $P_k = P(f_k)$ is wildly fluctuating.

Note: we did not do any average yet!
The Discrete Fourier Transform (DFT)

Properties of the DFT

DFT-Specific Properties

Power spectrum estimate

Periodogram statistics

Theorem

If \( g \) is a discrete 2-order stationary process, \( g = \mathcal{N}(0, \sigma^2_g) \), then

1. \( G = D[g] = \mathcal{N}(0, \frac{1}{N} \sigma^2_g) \) (\( \mu_G = 0; \sigma^2_G = \frac{1}{N} \sigma^2_g \));
2. \( P_k \) and \( P_{k+m} \) are independent for \( m \neq 0 \), and

\[
\sigma^2_P = 4\sigma^4_g, \quad \text{for} \quad n \neq 0.
\]

- Increasing \( N \) increases frequency resolution but
  - does not reduce oscillations: \( P_k \) are independent
  - does not reduce errors: \( \sigma^2_P \) independent of \( N \).

Heuristical explanation

Problem:

1. Error (variance) of auto-correlation \( \{c_n\} \) decays like \( 1/N \) as \( N \to \infty \).
2. \( \{P_k\} = D[\{c_n\}] = \) linear combination of \( N \) auto-correlations
3. cumulative effects of \( N \) \( 1/N \)-like terms is order 1.

Fix:

Since the \( c_n \to 0 \) as \( n \to \infty \), end points contribute only error!

\( \Rightarrow \) **Truncate the auto-correlation (maximum lag)**

\( \Rightarrow \) **Window the time series!**

\( \Rightarrow \) **Smooth the periodogram!**
The Discrete Fourier Transform (DFT)

Properties of the DFT

DFT-Specific Properties

Power spectrum estimate

The periodogram

Consistent estimates. Spectral windows

Bartlett’s smoothed estimator and window

Spectral estimation by averaging over time blocks

Discarding large lags ⇔ Windowing

Typical autocorrelation (blue line) curve and associated variance
(read: error – bars and light blue band).

Excluding large lags ⇔ windowing the time series by a window of \( \Delta T \) length.

Bartlett procedure: average over time blocks

Bartlett: goal is to reduce variance

1. Split \( \{g_n\} \), \( n = 0, 1, \ldots, N - 1 \) into \( M \) sub-blocks of \( \{g_n\}_m \) of \( N/M \) points.
2. Calculate autocovariance \( \{c_n\}_m \) for each sub-block \( m = 0, \ldots, M \).
3. Calculate periodogram \( \{P_k\}_m = D[\{c_n\}_m] \) for each sub-block.
4. Average periodogram over sub-blocks: \( \{H_k\} = \frac{1}{M} \sum_m \{P_k\}_m \).

The procedure is equivalent to windowing by triangular (Bartlett) window.
Welch’s direct method

Compute DFT of the time series rather than DFT of the autocovariance:

1. Divide \( \{g_n\} \) into \( M \) sub-blocks of \( \{g_n\}_m \) of \( K = N/M \) points; sub-blocks allowed to overlap (typically by 50%).
2. Window each sub-block: \( \{g_n^W\} = \{g_n\}_m \{w_n\}, \{w_n\} = \text{window} \). Redundancy due to overlapping is reduced by window tapering.
3. Tapering reduces the variance (see next slide).
4. Adjust for variance change due to windowing:
   \[
   \{g_n^W\}_m \rightarrow \{g_n^W\}_m \times \sqrt{\sum \{g_n\}_m^2} / \sqrt{\sum \{g_n^W\}_m^2}.
   \]
5. DFT each tapered sub-block, calculate the periodogram:
   \[
   \{P_k\}_m = K |G_k^W|^2.
   \]
6. Average over sub-blocks:
   \[
   \{H_k\} = \frac{1}{M} \sum_m \{P_k\}_m.
   \]

Tapering reduces variance

Windowed signal (red) has smaller variance than the original signal (blue).
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The periodogram
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Spectral estimation by averaging over time blocks

Spectrum

\[ \{P_k\} = N |G_k|^2, \text{ calculated using the entire time series}; \]

b) Spectrum estimate by Welch’s direct method.

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Practical Spectral Analysis