I) Show that the following properties of a Cumulative Distribution Function (CDF) follow from the definition of the CDF and the properties of probability:

(1) $F(-\infty) = 0$ and $F(+\infty) = 1$.
(2) $F$ is a non-decreasing function, i.e. if $x_1 \leq x_2$, then $F(x_1) \leq F(x_2)$.
(3) If $F(x_0) = 0$, then $F(x) = 0$ for every $x \leq x_0$.
(4) $P\{X > x\} = 1 - F(x)$.
(5) $P\{x_1 < X \leq x_2\} = F(x_2) - F(x_1)$.

**Proof. Property 1:** By definition, $F_X(a) = P(A)$ where $A = \{X \leq a\}$. Then $F(= +\infty) = P(A)$, with $A = \{X \leq +\infty\} = \{X \in \mathbb{R}\} = S$. Therefore $F(= +\infty) = P(S) = 1$. Similarly, Then $F(= -\infty) = P(A)$, with $A = \{X \leq -\infty\}$. This event is the impossible event, RV $X$ is a real variable, so $F(= -\infty) = P(\emptyset) = 0$. □

**Proof. Property 2:**
This is a consequence of the fact that probability is a nondecreasing function, i.e. if $A \subseteq B$ then $P(A) \leq P(B)$. Indeed, recall that if $A \subseteq B$ then the set $B$ can be partitioned into $B = A + (B \cap \bar{A})$, so, using the additivity property of probability, $P(B) = P(A) + P(B \cap \bar{A}) \geq P(A)$. If $x_1 \leq x_2$, then the corresponding events $A = \{X \leq x_1\}$ and $B = \{X \leq x_2\}$ have the property that $A \subseteq B$. Therefore $P(\{X \leq x_1\}) \leq P(\{X \leq x_2\})$, which by the definition of a CDF means $F(x_1) \leq F(x_2)$. □

**Proof. Property 3:**
This property is related to the nondecreasing property of the CDF: going backwards, to the left, toward $-\infty$, the CDF can only decrease or stay constant. If $F(x_0) = 0$, to the left of $x_0$ it can only stay constant (=0) since it cannot take negative values.
For $x \leq x_0$ the events $A = \{X \leq x\}$ and $B = \{X \leq x_0\}$ have the property that $A \subseteq B$, therefore

$$0 \leq F(x) = P(\{X \leq x\}) \leq P(\{X \leq x_0\}) = F(x_0) = 0$$

therefore $F(x) = 0$. □

**Proof. Property 4:**
This is a consequence of the properties of probability of complementary events: recall that $P(\bar{A}) = 1 - P(A)$. Set $A = \{X \leq x\}$; then its complement (the set of all elementary events which are not in $A$) is $\bar{A} = \{X > x\}$, i.e. $\{X \leq x\} + \{X > x\} = S = \mathbb{R}$. As above, $P(\bar{A}) = 1 - P(A)$, but with from the definition of the CDF, $P(\bar{A}) = P(\{X > x\}) = 1 - P(\{X \leq x\}) = 1 - F(x)$. □
**Proof. Property 5:**

This property shows how to use the CDF to calculate the probability that RV $X$ returns a value in a given interval $[x_1, x_2]$. The CDF is defined on semi-infinite intervals of the form $(-\infty, x_1]$ and $(-\infty, x_2]$. Note that the events $A = \{X \in (-\infty, x_1]\}$, $B = \{X \in (-\infty, x_2]\}$ and $C = \{X \in (x_1, x_2]\}$ are in the relationship $B = A + C$ since $A$ and $C$ are disjoint and $B = A \cup C$. Therefore $P(B) = P(A) + P(C)$, or $P(C) = P(B) - P(A)$. Hence the property. □

II) The number $x_u$ for which the CDF $F(x)$ has the value $F(x_u) = u$ is called the $u$-percentile of $F$. Show that if the derivative $f(x) = \frac{dF}{dx}$ is even, that is $f(-x) = f(x)$, then

1) $F(-x) = 1 - F(x)$
2) $x_{1-u} = -x_u$.
3) Draw a sketch of the CDF and its derivative. Explain on the plot the above relations.

**Proof.**

1) Since $f(x) = \frac{dF}{dx}$, one can write

$$F(x) = \int_{-\infty}^{x} f(s) \, ds,$$

and

$$F(-x) = \int_{-\infty}^{-x} f(s) \, ds.$$

Changing the variable in the last integral to $t = -s$, with

$$s = -t; \quad ds = -dt; \quad s = -x \mapsto t = x; \quad s = -\infty \mapsto t = \infty;$$

and using the fact that $f(-t) = f(t)$ yields

$$\int_{-\infty}^{-x} f(s) \, ds = \int_{\infty}^{x} f(-t) \, (-dt) = \int_{x}^{\infty} f(-t) \, dt = \int_{x}^{\infty} f(t) \, dt = F(\infty) - F(x) = 1 - F(x).$$

2) Using the above definition of the percentile, $F(x_{1-u}) = 1 - u$ and $F(x_u) = u$. Obviously, this means that $F(x_{1-u}) = 1 - F(x_u)$. From point 1), this implies that $x_{1-u} = -x_u$.

3) A sketch is shown below
III) Show that if $X$ is an RV that can take values only in a finite interval, $a \leq X \leq b$, then $F(x) = 1$ for every $x \geq b$ and $F(x) = 0$ for every $x \leq a$.

Proof. If $a \leq X \leq b$, from the definition of a CDF, $F(a) = 0$ and $F(b) = 1$. Since $F$ is nondecreasing with values in the interval $[0, 1]$, from $x \geq b$ follows that $1 = F(b) \leq F(x) \leq 1$, therefore $F(x) = 1$ for any $x \geq b$. A similar reasoning works for the second statement: if $x \leq a$,

$$0 \leq F(x) \leq F(a) = 0.$$  

\[\square\]

IV) Show that if $X$ and $Y$ are two RVs such that $X \leq Y$ at every trial, then $F_Y(a) \leq F_X(a)$ for every $a \in \mathbb{R}$.

Proof. From the hypothesis, $X$ and $Y$ are two RVs such that $X \leq Y$ at every trial, i.e. for every elementary event,

$$X(\zeta) \leq Y(\zeta).$$

Let $A_X$ be the set of elementary events $\zeta$ such that the observed value $X(\zeta) \leq a$. In set algebra notation $A_X = \{ \zeta : X(\zeta) \leq a \}$. Denote similarly, $A_Y = \{ \zeta : Y(\zeta) \leq a \}$.

By the definition of CDF,

$$F_X(a) = P(A_X); \quad F_Y(a) = P(A_Y).$$

If $A_Y \subseteq A_X$, then using the “monotonicity” property of probability, $P(A_Y) \leq P(A_X)$, and consequently $F_Y(a) \leq F_X(a)$.

Indeed, let $\zeta \in A_Y$. This means that $Y(\zeta) \leq a$, but also since $X(\zeta) \leq Y(\zeta) \leq a$, so $\zeta$ also belongs to $A_X$, $\zeta \in A_X$.  

\[\square\]

V) RV $X$ takes only positive values. Show that if the conditional probability

$$P(\{s < X \leq s + s_1\} | \{X > s\}) = P(\{X \leq s_1\})$$

for any $s, s_1 \geq 0$, then the Cumulative Distribution Function (CDF) $F_X(t) = 1 - e^{-ct}$, where $c$ is a constant.
Proof. Denote $\mathcal{A} = \{ s < X \leq s + s_1 \}$, $\mathcal{B} = \{ X > s \}$ and $\mathcal{C} = \{ X \leq s_1 \}$; with the definition of conditional probability, the relationship can be written

$$P(\mathcal{A}|\mathcal{B}) = \frac{P(\mathcal{A} \cap \mathcal{B})}{P(\mathcal{B})} = P(\mathcal{C}).$$

Note also that $\mathcal{A} \subset \mathcal{B}$, which means that $\mathcal{A} \cap \mathcal{B} = \mathcal{A} = \{ s < X \leq s + s_1 \}$, so in fact the relation is

$$\frac{P(\mathcal{A})}{P(\mathcal{B})} = P(\mathcal{C}).$$

Denote by $F_X(t)$ the CDF of RV $X$. For the sake of clarity, let’s rename $s_1 = \Delta s$. Then

$$P(\mathcal{A}) = F(s + \Delta s) - F(s); \quad P(\mathcal{B}) = 1 - F(s); \quad P(\mathcal{C}) = F(\Delta s).$$

The relation becomes

$$\frac{F(s + \Delta s) - F(s)}{1 - F(s)} = F(\Delta s).$$

valid for any $s, \Delta s \geq 0$. For a fixed $s$, let $\Delta s \to ds$. Then

$$F(\Delta s) \to F(ds) = F(0) + \left( \frac{dF}{ds} \right)_0 ds,$$

$$F(s + \Delta s) - F(s) \to F(s + ds) - F(s) = \frac{dF}{ds} ds.$$

Note also that $F(0) = 0$. The relation then becomes

$$\frac{1}{1 - F(s)} \frac{dF}{ds} ds = \left( \frac{dF}{ds} \right)_0 ds$$

or equivalently

$$\frac{F'}{1 - F} = c, \quad \text{where} \quad c = \left( \frac{dF}{ds} \right)_0.$$  

This is a differential equation; integrating from $s = 0$:

$$\frac{dF}{1 - F} = c \, ds, \quad \int_0^F \frac{dF}{1 - F} = cs; \quad - \int_0^F \frac{d(1 - F)}{1 - F} = cs.$$

so

$$F(s) = 1 - e^{-cs}.$$  

□