I) If events $A$ and $B$ are independent, then events $\bar{A}$ and $B$ are independent, and events $\bar{A}$ and $\bar{B}$ are also independent.

**Proof.** By definition, $A$ and $B$ independent $\iff P(A \cap B) = P(A)P(B)$. But $B = (A \cap B) + (\bar{A} \cup B)$, so $P(B) = P(A \cap B) + P(\bar{A} \cup B)$, which yields $P(\bar{A} \cap B) = P(B) - P(A \cap B) = P(B) - P(A)P(B) = P(B)[1 - P(A)] = P(B)P(\bar{A})$. Repeat the argument for the events $\bar{A}$ and $\bar{B}$, this time starting from the statement that $\bar{A}$ and $B$ are independent and taking the complement of $B$. \qed

II) If $A \subset B$, $P(A) = 1/4$, and $P(B) = 1/3$, find $P(A | B)$ and $P(B | A)$.

**Proof.** (solution) If $A \subset B$, then $A \cap B = A$. Then, using the definition, $P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{3}{4}$. Then also $P(B | A) = \frac{P(A)}{P(A)} = 1$. \qed

III) Three events $A_j$, $j = 1, 2, 3$ are called independent if

1. the are independent in pairs $P(A_j \cap A_l) = P(A_j)P(A_l)$ for any $j \neq l$

2. and $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

(Caution: 3 events might be independent in pairs but not independent), Show that

1: Any one event is independent of the intersection of the other two.

**Proof.** If events $A_j$, $j = 1, 2, 3$ are independent, by condition 2) in the definition above $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$. Also, by condition 1) in the definition above, for example $P(A_1)P(A_2) = P(A_1 \cap A_2)$ so the right-hand side can be written as $P(A_1 \cap A_2 \cap A_3) = P(A_1 \cap A_2)P(A_3)$. Denote $B = (A_1 \cap A_2)$. Then $P[(A_1 \cap A_2) \cap A_3] = P(B \cap A_3) = P(B)P(A_3)$, so $B = (A_1 \cap A_2)$ and $A_3$ are independent. \qed

2: Replacing one or more of these events with complements the resulting three events are still independent.
**Proof.** Replace for example $A_3 \mapsto \bar{A}_3$. From Exercise 1, we know that the complement of any of the events is independent of the rest of the two events. This takes care of Condition 1 in the 3-event mutual independence. We have to show that Condition 2 is also satisfied, that is,

$$P(A_1 \cap A_2 \cap \bar{A}_3) = P(A_1)P(A_2)P(\bar{A}_3).$$

Denote $B = A_1 \cap A_2$ and partition it using $A_3$,

$$B = (B \cap A_3) + (B \cap \bar{A}_3),$$

Using the additivity of probability, thsi means that

$$P(B) = P(B \cap A_3) + P(B \cap \bar{A}_3),$$

that is

$$P(B \cap \bar{A}_3) = P(B) - P(B \cap A_3),$$

Replacing $B = A_1 \cap A_2$ and with $A_j, j = 1, 2, 3$ are independent, yields

$$P(A_1 \cap A_2 \cap \bar{A}_3) = P(A_1 \cap A_2) - P(A_1 \cap A_2 \cap A_3) = P(A_1 \cap A_2) - P(A_1 \cap A_2)P(A_3),$$

or

$$P(A_1 \cap A_2 \cap \bar{A}_3) = P(A_1 \cap A_2) - P(A_1 \cap A_2)P(A_3) = P(A_1 \cap A_2) [1 - P(A_3)].$$

Hence

$$P(A_1 \cap A_2 \cap \bar{A}_3) = P(A_1 \cap A_2)P(\bar{A}_3) = P(A_1)P(A_2)P(\bar{A}_3).$$

□

**IV) (Total Probability)** Show that if the events $A_j, j = 1, 2, \ldots, N$ are a partition of the probability space $S$, i.e.

1. they are mutually exclusive, $A_j \cap A_l = \emptyset$ for any $j \neq l$
2. their union is the entire space, $\bigcup_{j=1}^N A_j = S$,

then the probability of an event $B$ may be written as

$$P(B) = P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + \ldots + P(B \mid A_N)P(A_N).$$
Proof. We have seen that if \( A \subset S \), the sets \( A \) and \( \bar{A} \) form a partition of \( S \), \( S = A + \bar{A} \), and in addition, any set \( B \) can be partitioned into

\[
B = (B \cap A_1) + (B \cap \bar{A}).
\]

This can be generalized for any partition of \( S \); if \( A_j, j = 1, 2, \ldots, N \) are a partition of \( S \),

\[
B = (B \cap A_1) + (B \cap A_2) + \ldots + (B \cap A_N).
\]

This is intuitively true: the sets \( A_j, j = 1, 2, \ldots, N \) cover the entire space and \( B \) will be expressed as a union of its intersections with \( A_j, j = 1, 2, \ldots, N \). Since \( A_j, j = 1, 2, \ldots, N \) are disjoint, the intersections with \( B \) are also disjoint, hence the result.

Applying the additivity property of probability yields

\[
P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots + P(B \cap A_N).
\]

From the definition of conditional probability,

\[
P(B | A_1) = \frac{P(B \cap A_1)}{P(A_1)} \Rightarrow P(B \cap A_1) = P(B | A_1) P(A_1),
\]

hence the result.

(A bit more convincing – not necessary for the homework), to show that if \( A_j, j = 1, 2, \ldots, N \) are a partition of \( S \),

\[
B = (B \cap A_1) + (B \cap A_2) + \ldots + (B \cap A_N),
\]

we need to show that

1) \( B \subset (B \cap A_1) + (B \cap A_2) + \ldots + (B \cap A_N) \), and

2) \( (B \cap A_1) + (B \cap A_2) + \ldots + (B \cap A_N) \subset B \).

Obviously, \( B \cap A_j, j = 1, 2, \ldots, N \) are mutually exclusive (disjoint).

1. Assume that \( \zeta \in B \subset S \) (\( \zeta \) is an element of \( B \), \( B \) is a subset of \( S \)). Then \( \zeta \in S \), but since \( A_j, j = 1, 2, \ldots, N \) are a partition of \( S \), \( \zeta \) must be in one of them, i.e. there is a \( j \) such that that \( \zeta \in A_j \). Hence there is a \( j \) such that \( \zeta \in B \cap A_j \).

2. Reciprocally, assume that \( \zeta \in (B \cap A_1) + (B \cap A_2) + \ldots + (B \cap A_N) \). Since \( B \cap A_j \), \( j = 1, 2, \ldots, N \) are mutually exclusive (disjoint) there is a \( j \) such that \( \zeta \in B \cap A_j \). This means that \( \zeta \in B \). \qed

V) Show that a set \( S \) with \( n \) elements has \( 2^n \) subsets and

\[
\frac{n(n-1)\ldots(n-k+1)}{1\cdot2\cdot\ldots\cdot k} = \frac{n!}{k!(n-k)!}
\]

subsets with \( k \) elements.
Proof. 1. The easiest way to show this is by induction. Start with a set with \( n = 1 \) elements, \( S_1 = \{\zeta_1\} \). The subsets are listed below and their number is indeed \( 2^1 \).

\[
\emptyset \quad \{\zeta_1\}.
\]

Now let \( S_2 = \{\zeta_1, \zeta_2\} \) have \( n = 2 \) elements. Its subsets are indeed 4, as shown below:

\[
\emptyset \quad \{\zeta_1\} \quad \{\zeta_2\} \quad \{\zeta_1, \zeta_2\}.
\]

Notice that the subsets are formed from the subsets of \( S = \{\zeta_1\} \), by “adding” the new element. The subsets of \( S_1 = \{\zeta_1\} \) in the first row are , the second row are

\[
\emptyset \quad \{\zeta_1\} \quad \{\zeta_2\} \quad \{\zeta_1, \zeta_2\}.
\]

The number of subsets doubles. In general, \( S_{n+1} = S_n \cup \{\zeta_{n+1}\} \), so the subsets of \( S_{n+1} \) contain all the subsets of \( S_n \) plus all the “new” subsets, which contain also the element \( \zeta_{n+1} \), formed by the union of any of the subsets of \( S_n \) with the set \( \{\zeta_{n+1}\} \). Hence the number of subsets of \( S_{n+1} \) is \( 2^n + 2^n = 2 \cdot 2^n = 2^{n+1} \). \( \square \)