3. Fourier Analysis

Data Analysis Techniques in Oceanography OCP6168

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Definition

Fixed $\zeta = \zeta_0$: Function $X(t) = X(t, \zeta_0)$.

Fixed $t = t_0$: Random Variable $X(\zeta) = X(t_0, \zeta)$. 
Summary

Names and definitions given for 2RVs and 2 processes $X(t)$, $Y(t)$.
To obtain the definitions for “auto-” moments, set $Y = X$ in column 2.

<table>
<thead>
<tr>
<th>2 RV $X$, $Y$</th>
<th>2 Processes $X(t)$, $Y(t)$, (RV $X(t_1), Y(t_2)$)</th>
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<tbody>
<tr>
<td><strong>Correlation</strong> (Inner product) $R = E{XY}$</td>
<td><strong>Cross (auto)- correlation</strong> $R_{XY}(t_1, t_2) = E{X(t_1)Y(t_2)}$</td>
</tr>
<tr>
<td><strong>Covariance</strong> $C = E{(X - \mu_X)(Y - \mu_Y)^*}$</td>
<td><strong>Cross (auto)- covariance</strong> $C_{XY}(t_1, t_2) = E{(X - \mu_X)<em>{t_1}(Y - \mu_Y)</em>{t_2}^*}$</td>
</tr>
<tr>
<td><strong>Variance</strong> $\sigma^2 = E{</td>
<td>X - \mu_X</td>
</tr>
<tr>
<td><strong>Correlation coefficient</strong> $\rho = C_{xy}/\sigma_x\sigma_y$</td>
<td><strong>Correlation coefficient</strong> $\rho_{XY}(t_1, t_2) = C_{XY}(t_1, t_2)/\sigma_X(t_1)\sigma_Y(t_2)$</td>
</tr>
</tbody>
</table>

Measurement data

The auto-correlation $R(t_2, t_1)$ has a very specific structure.
(also auto-covariance, or correlation coefficient)
$\Delta t = t_2 - t_1 = \text{time lag}$.
Harmonic series

Definitions

- Harmonic process: $H(t) = A \cos(\omega t + T)$; $A$, $T$ mutually uncorrelated, $T$ uniformly distributed in
- Harmonic series: $H(t) = \sum_{n=1}^{N} A_n \cos(\omega_n t + T_n)$;

Mean: $\mu_H = 0$.
Variance: $\sigma^2_H = \sum_{n=1}^{N} \frac{1}{2} E\{A_n^2\} = \sum_{n=1}^{N} \sigma^2_n$

Auto-correlation:

- (auto-covariance) $R_{HH}(t_1, t_2) = \sum_{n=1}^{N} \frac{1}{2} E\{A_n^2\} \cos(\omega_n (t_1 - t_2))$,
- $R_{HH}(\Delta t) = \sum_{n=1}^{N} \frac{1}{2} E\{A_n^2\} \cos(\omega_n \Delta t)$.

Depend only on the “lag” $\Delta t$!!

Average Power: $R_{HH}(t) = \sum_{n=1}^{N} \frac{1}{2} E\{A_n^2\} = \sum_{n=1}^{N} \sigma^2_n = \sigma^2_H$.

2-order stationary processes

Definition

Process $X(t)$ is 2-order stationary if:

Mean: $\mu = E\{x(t)\} = \text{constant, independent of } t$,
Auto-correlation: $R(t_1, t_2) = R(t, \Delta t) = R(\Delta t)$ independent of $t$.
Average Power: $R(0) = E\{|X(t)|^2\} = \text{constant, independent of } t$,

- Auto-covariance: $C(\Delta t) = R(\Delta t) - |\mu|^2$ depends only on time lag.
- Correlation coefficient: $\rho(\Delta t) = \frac{C(\Delta t)}{C(0)}$ depends only on time lag.
Harmonic analysis

Harmonic series processes are 2-order stationary.
- Can any 2-order stationary process be represented as a harmonic series process?
- If not, then maybe:
  - Can some 2-order stationary process be represented as a harmonic series process?
  - What kind?
  - Can the representation be modified to an integral instead of a sum?
- Would such a representation be unique?

"Represent" = approximate arbitrary "close" (e.g. MS distance) a given process
"Series" = sum with an infinite number of terms...
"Integral" = a limiting process applied to a sum...
What would that mean in stochastic processes?

Basic idea

Functions here are deterministic, and periodic.
Real representation:

\[
X(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt), \quad \text{with} \quad a_n, b_n \in \mathbb{R}
\]

Complex representation:

\[
X(t) = \sum_{n=-\infty}^{\infty} c_n \exp int
\]

- What is so special about (\sin nt, \cos nt) or \exp int?
- Any 2\pi-periodic function = series of sinusoidals?
- What are the conditions \(X(t)\) has to satisfy?
Orthogonality of \( \sin, \cos, \exp \)

It might be possible:

\[
\begin{align*}
\int_{-\pi}^{\pi} \cos nt \cos mt &= 0 \quad \text{for } m \neq n \\
\int_{-\pi}^{\pi} \sin nt \sin mt &= 0 \quad \text{for } m \neq n \\
\int_{-\pi}^{\pi} \cos nt \sin mt &= 0 \quad \text{for all } m, n
\end{align*}
\]

and in complex:

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(int) \exp(-imt) \, dt = \delta_{nm} = \begin{cases} 
0 & n \neq m \\
1 & n = m
\end{cases}
\]

Scalar product! Maybe there is a vector space?

Then \( \sin, \cos, \exp \) might be a basis.

---

**Fourier-Euler Formulae**

**Real Fourier series**

If \( X(t) \) is a deterministic, \( 2\pi \)-periodic function, seek a representation

\[
X(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt), \quad \text{with } a_n, b_n \in \mathbb{R}
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} X(t) \cos nt \, dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} X(t) \sin nt \, dt.
\]

**Complex Fourier Series**

\[
X(t) = \sum_{n=-\infty}^{\infty} c_n \exp(int), \quad \text{with } c_n \in \mathbb{C}, \quad \text{and } c_{-n} = c_n^*.
\]

\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(t) \exp(-int) \, dt.
\]
The vector space: Class $L^2(-\pi, \pi)$

To admit a Fourier series representation, $\int_{-\pi}^{\pi} |X(t)|^2 \, dt < \infty$.
The space is called the $L^2(-\pi, \pi)$ class.

Definitions

1. $L^2(-\pi, \pi) = \infty$-Dim. vector space + inner product = Hilbert space.
2. If $X(t)$ and $Y(t)$ are two functions of class $L^2(-\pi, \pi)$, the inner product
   \[ \langle X, Y \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(t) Y^*(t) \, dt. \]
3. The norm in $L^2(-\pi, \pi)$:
   \[ \|X\|^2 = \langle X, X \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(t)|^2 \, dt. \]

Class $L^2(-\pi, \pi)$ as a vector space

- $L^2(-\pi, \pi)$ class functions $X(t)$ are vectors;
- Set of vectors $\{U_n\} = \{\exp i n t\}$ is a basis.
- The unique (Fourier) representation = decomposition on the basis
  \[ X = \sum_{n=-\infty}^{\infty} c_n U_n; \]
- $c_n$ are the components (projections) of $X$ in this basis;
- Length of a vector (norm): $X = \sum x_n U_n$ and $Y = \sum y_n U_n$,
  \[ \|X\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 \|U_n\|^2 \quad \text{and} \quad \langle X, Y \rangle = \sum_{n=-\infty}^{\infty} x_n y^*_n \|U_n\|^2. \]
- If basis is orthonormal ($\|U_n\| = 1$), then
  \[ \langle X, Y \rangle = \sum_{n=-\infty}^{\infty} x_n y^*_n \quad \text{and} \quad \|X\|^2 = \sum_{n=-\infty}^{\infty} |x_n|^2. \]
**General Periodicity**

To generalize from $2\pi$-periodic functions to $T$-periodic, $T \in \mathbb{R}$.

- Change variables $s = t \frac{T}{2\pi}$, and $t = \frac{2\pi}{T} s$,
- Define frequency $f_n = \frac{n}{T}$, radian frequency $\omega_n = \frac{2\pi}{T}$:

### General-periodicity Fourier-Euler relations

$$X(t) = \sum_{n=-\infty}^{\infty} c_n \exp(2\pi if_n t), \quad c_n = \frac{1}{T} \int_{-T/2}^{T/2} X(t) \exp(-2\pi if_n t) \, dt,$$

**Inner product**

$$\langle X, Y \rangle = \frac{1}{T} \int_{-T/2}^{T/2} X(t) Y^*(t) \, dt.$$

**Properties of Fourier Coefficients**

Let

$$X = \sum x_n U_n; \quad Y = \sum y_n U_n; \quad Z = \sum z_n U_n$$

with $U_n = \exp(i\omega_n t)$, with $\omega_n = 2\pi n/T$;

- **Linearity** If $Z(t) = aX(t) + bY(t)$, then $z_n = ax_n + by_n$.
  - If $X$ and $Y$ have Fourier representation so does $Z$.
- **Even function** $X$ is even $\Rightarrow c_n$ are all real, $c_{-n} = c_n$.
- **Odd function** $X$ is odd $\Rightarrow c_n$ are all imaginary.
Parseval Relation

If \( X = \sum_n x_n U_n \) (\( X \) is \( L^2 \) class, \( U_n = \exp 2\pi i f_n t \))

\[
\|X\|^2 = \sum_{n=\infty}^{\infty} |x_n|^2.
\]

Remember stochastic processes?

- Total power, 2-order stationary process \( X(t) \): \( \sigma_X^2 = R(0) = \mathbb{E}\{|X(t)|^2\} \).
- Total power, harmonic process \( X(t) \): \( \sigma_H^2 = R(0) = \mathbb{E}\{|H(t)|^2\} = \sum \sigma_n^2 \).
- Calculate the mean as time average:
  \[
  R(0) = \int_a^b |X(t)|^2 \, dt.
  \]

Spectral Analysis of Periodic Functions

Total energy

\[
total\ energy = \int_{-T/2}^{T/2} |X(t)|^2 \, dt = T \sum_{n=-\infty}^{\infty} |c_n|^2
\]

Total power

\[
total\ power = total\ energy / period = \frac{1}{T} \int_{-T/2}^{T/2} |X(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |c_n|^2
\]

Single-frequency oscillation \( X(t) = c_n \exp(2\pi f_n t) \), with \( f = n/T \),

Total power \( = |c_n|^2 \)

Spectral representation Fourier-Euler relations for (periodic) function \( X \).

Power spectrum (Parseval) Total power = sum of individual harmonic power.
Example: ramp function.

\[ X(t) = t, \text{ with } t \in [-T/2, T/2]. \]
\[ X(-t) = X(t), \text{ } X \text{ is odd:} \]
\[ c_n = \frac{1}{T} \int_{-T/2}^{T/2} t \exp(-i\omega_n t) \, dt \Rightarrow c_n = \frac{(-1)^n i}{2\pi f_n}; \quad |c_n| = \frac{1}{2\pi f_n}; \quad i = \exp \left(\frac{i\pi}{2}\right); \]

\[ X(t) = \sum_{n=-\infty}^{\infty} c_n \exp(2\pi i f_n t); \quad c_n = \frac{1}{T} \int_{-T/2}^{T/2} X(t) \exp(-2\pi i f_n t) \, dt \]

Substitute \( c_n \), define \( \Delta f = f_n - f_{n-1} = \frac{1}{T} \),
\[ X(t) = \sum_{n=-\infty}^{\infty} \left[ \int_{-T/2}^{T/2} X(t) \exp(-2\pi i f_n t) \, dt \right] \exp(2\pi i f_n t) \Delta f \]

Let \( T \to \infty, \Delta f = \frac{1}{T} \to 0 \) and \( \sum \to \int \)
\[ X(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(t) \exp(-2\pi i f_n t) \, dt \exp(2\pi i f_n t) \, df \]
Fourier Analysis
Linear transformations and filters
DETERMINISTIC PERIODIC Functions: Fourier Series
DETERMINISTIC APERIODIC Functions: Fourier Integrals
2-Order Stationary Random Process

Fourier Pair

Fourier Transform Pair

\[ \hat{X}(f) = \int_{-\infty}^{\infty} X(t) \exp(-2\pi ift) \, dt \quad \text{direct FT} \]
\[ X(t) = \int_{-\infty}^{\infty} \hat{X}(f) \exp(2\pi ift) \, df \quad \text{inverse FT} \]

- \( \hat{X}(f) \) is the Fourier transform of \( X(t) \);
- \( \hat{X}(f) \) and \( X(t) \) are called a Fourier pair;
- Fourier Transform operator: \( F[\cdot] = \int_{-\infty}^{\infty} \cdot \exp(-2\pi ift) \, dt \)
- Inverse Fourier Transform operator: \( F^{-1}[\cdot] = \int_{-\infty}^{\infty} \cdot \exp(2\pi ift) \, df \)
- \( \hat{X}(f) = F[X(t)] \), \( X(t) = F^{-1} \hat{X}(f) \);
- Dual nature of FT

Existence of Fourier Transform Pair

Existence of Fourier pair \( X(t) \), \( \hat{X}(f) \) more problematic than Fourier series

1. If \( X(t) \) is such that \( \int_{-\infty}^{\infty} |X(t)| \, dt < \infty \), it has a FT \( \hat{X}(f) \).
2. Inverse \( \hat{X}(f) \) exists if additional conditions of “good behavior” for \( X(t) \) are imposed (e.g. \( X(t) \to 0 \) as \( t \to \pm \infty \))
Existence of Fourier Transform Pair

However:

$$\exp(2\pi if_0 t) = \int_{-\infty}^{\infty} \delta(f - f_0) \exp(2\pi if t) \, df,$$

- $\delta(f)$ and $\exp(2\pi if t)$ are like a Fourier pair.
- Generalization of the Fourier integral concept?
- generalized functions (also called distributions).

Properties of FT (1)

Let $X(t)$ and $\hat{X}(f) = F[X(t)]$ be a FT pair.

- **Linearity:** $F[aX + bY] = aF[X] + bF[Y]$.
- **Shift 1:** $F[X(t - a)] = \hat{X}(f) \exp(-2\pi if a)$.
- **Shift 2:** $F[X(t) \exp(2\pi if_0 t)] = \hat{X}(f - f_0)$.
  (Duality: Shift in argument $\rightarrow$ oscillation of the transform)
- **Scale:** $F[X(at)] = \frac{1}{|a|} \hat{X}\left(\frac{f}{a}\right)$ $\Rightarrow$ Consequence: $F[X(-t)] = \hat{X}(-f)$.
  (Duality: stretching $t \rightarrow$ compression in $f$)
Properties of FT (2)

Derivative 1: \( \frac{d^n}{df^n} \hat{X}(f) = (-2\pi i)^n F[t^n X(t)]. \)

Derivative 2: \( F \left[ \frac{d^n}{dt^n} X(t) \right] = (-2\pi i f)^n \hat{X}(f). \)

Double F: \( F[F[X(t)]] = X(-t). \)

Duality: \( F[X(t)] = \hat{X}(f), \) and \( F[\hat{X}(f)] = X(-t). \)

Properties of FT (3)

- Symmetry of the function and its FT

<table>
<thead>
<tr>
<th>( X(t) )</th>
<th>( \hat{X}(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>real, even</td>
<td>real, even</td>
</tr>
<tr>
<td>real, odd</td>
<td>imaginary, odd</td>
</tr>
<tr>
<td>imaginary, even</td>
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</tr>
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<td>imaginary, odd</td>
<td>real, odd</td>
</tr>
</tbody>
</table>
Fourier Analysis
Linear transformations and filters

DETERMINISTIC PERIODIC Functions: Fourier Series
DETERMINISTIC APERIODIC Functions: Fourier Integrals

Examples: Rectangular pulse

\[ X(t) = \begin{cases} 
1 & |t| \leq a \\
0 & \text{otherwise} 
\end{cases} ; \quad \hat{X}(f) = \frac{2a \sin(2\pi fa)}{2\pi fa} = 2a \text{sinc}(2\pi fa). \]

![Graph of rectangular pulse](a)

![Graph of Fourier transform](b)

Examples: Gaussian

The Gaussian is invariant to the FT

\[ F \left[ \exp \left( -\pi t^2 \right) \right] = \exp \left( -\pi f^2 \right). \]

Symmetry examples based on the Gaussian: (real - imaginary)
Examples: shifted Gaussian

Think about a wave group.

IFT of a Gaussian at $f_0$, width $\sigma$:

$$X(t) = A \exp(-2\pi^2 \sigma^2 t) \exp(2\pi if_0 t).$$

“Length” $\sigma_t$ of group inversely proportional to spectral width $\sigma_f$

$$\sigma_t \sim \frac{1}{\sigma_f}$$

Convolution and correlation

Definitions

- Convolution product: $Z(t) = (X \circ Y)(t) = \int_{-\infty}^{\infty} X(\xi) Y(t-\xi) \, d\xi$.
- Correlation product: $Z(t) = (X \star Y)(t) = \int_{-\infty}^{\infty} X(\xi) Y^*(t+\xi) \, d\xi$

Theorem

1. Convolution:
   a) $F[X \cdot Y] = F[X] \cdot F[Y]$
   b) $F[X \circ Y] = F[X] \circ F[Y]$

2. Correlation:
   a) $F[X \star Y] = F[X] \circ (F[Y])^*$
   b) $F[X \star X] = |F[X]|^2$

2b) is a significant result!
Convolution properties

1. Another way to write it:

\[(X \circ Y)(t) = \int_{-\infty}^{\infty} X(\xi_1)Y(\xi_2) \delta(t - \xi_1 - \xi_2) \, d\xi_1 d\xi_2.\]

2. Product-like properties
   - Commutativity: \(X \circ Y = Y \circ X\),
   - Associativity: \(X \circ (Y \circ Z) = (X \circ Y) \circ Z\),
   - Distributivity (addition): \(X \circ (Y + Z) = (X \circ Y) + (X \circ Z)\);

3. Multiplication with a constant \(a: a(X \circ Y) = aX \circ Y = X \circ aY\),

4. Derivative:
   \[\frac{d}{dt} (X \circ Y) = \frac{dX}{dt} \circ Y = X \circ \frac{dY}{dt},\]

5. Integral:
   \[\int_{-\infty}^{\infty} (X \circ Y) \, dt = \left(\int_{-\infty}^{\infty} X(t) \, dt\right) \left(\int_{-\infty}^{\infty} Y(t) \, dt\right).\]

Parseval relation, power spectrum

Parseval relation:

- Total energy:
  \[\int_{-\infty}^{\infty} |X(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{X}(f)|^2 \, df,\]

- Energy spectrum:
  \[|\hat{X}(f)|^2,\]

- Total energy in interval \([f, f + df]:\)
  \[|\hat{X}(f)|^2 \, df.\]

Fourier transform \(\sim\) superposition of harmonics \(f\) with energy \(|\hat{X}(f)|^2 \, df\).
Correlation and power spectrum (deterministic functions)

**Stochastic process**: auto-correlation is defined as

\[ R(\Delta t) = \mathbb{E} \{ X(t)X^*(t+\Delta t) \} = \int_{-\infty}^{\infty} X(t)X^*(t+\Delta t) \, dt \]

**Deterministic function**: Replace \( \mathbb{E} \{ \cdot \} \rightarrow \int \frac{\partial}{\partial t} \Rightarrow \) autocorrelation

\[ R(\Delta t) = \int_{-\infty}^{\infty} X(t)X^*(t+\Delta t) \, dt. \]

The Fourier transform of the autocorrelation is the energy spectrum:

\[
F[R(\tau)] = \int_{-\infty}^{\infty} R(\tau) e^{-2\pi i f \tau} \, d\tau = |\hat{X}(f)|^2.
\]

**Uncertainty Principle**

**Question**: “well localized in time” and “well localized in frequency”?

- Transform of a sinusoidal:
  \[ F[\exp(2\pi i f_0 t)] = \delta(f - f_0). \]

- Transform of a Gaussian:
  \[ F \left[ \frac{A}{\sqrt{2\pi\sigma_t^2}} \exp \left( -\frac{t^2}{2\sigma_t^2} \right) \right] = A \exp \left( -\frac{f^2}{2\sigma_f^2} \right); \quad \text{with} \quad \sigma_t = \frac{1}{2\pi\sigma_f} \]

One cannot reduce simultaneously the \( \sigma_f \) and \( \sigma_t \) (\( \sigma_f\sigma_t = 1 \)).
Heisenberg Theorem

Define mean “spread” $\sigma_t$ for $X$ (similar construction for $\sigma_f$ using $\hat{X}$):

1. Assume that $\|X\|^2 = \int_{-\infty}^{\infty} |X(t)|^2 dt < \infty$;

2. $F_T(t) = \frac{|X(t)|^2}{\|X\|^2} \geq 0$; $\int_{-\infty}^{\infty} F_T(t) dt = 1 \Rightarrow$ PDF of “time” RV $T$;

3. Following the definitions

$$
\mu_T = \int_{-\infty}^{\infty} t F_T(t) dt, \quad \sigma_T^2 = \int_{-\infty}^{\infty} (t - \mu_T)^2 F_T(t) dt.
$$

Heisenberg Uncertainty Principle: $\sigma_t^2 \sigma_f^2 \geq \frac{1}{(2\pi)^2}$ (for Gaussian)

No pair $X-\hat{X}$ is such that both cancel outside an interval.

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Fourier Analysis (review)

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<td>$T$-Periodic</td>
<td>$\int_{-\infty}^{\infty}</td>
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<tr>
<td></td>
<td>$\int_{-T/2}^{T/2}</td>
<td>X(t)</td>
</tr>
<tr>
<td>Synthesis</td>
<td>$X(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n f t}$</td>
<td>$X(t) = \int_{-\infty}^{\infty} \hat{X}(f) e^{2\pi i n f t} df$</td>
</tr>
<tr>
<td>Analysis</td>
<td>$c_n = \int_{-\infty}^{\infty} X(t) e^{-2\pi i n f t} dt$</td>
<td>$\hat{X}(f) = \int_{-\infty}^{\infty} X(t) e^{-2\pi i n f t} dt$</td>
</tr>
<tr>
<td>Energy spectrum</td>
<td>$</td>
<td>c_n</td>
</tr>
<tr>
<td>Total energy</td>
<td>$\sum_{n=-\infty}^{\infty}</td>
<td>c_n</td>
</tr>
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### Fourier Analysis (review)

#### Fourier Transform Properties

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<th>Formula</th>
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<td>Convolution</td>
<td>((X \circ Y)(t) = \int_{-\infty}^{\infty} X(\xi) Y(\xi - t) d\xi)</td>
</tr>
<tr>
<td></td>
<td>(X \circ Y = \hat{X} \circ \hat{Y})</td>
</tr>
<tr>
<td>Correlation</td>
<td>((X \star Y)(t) = \int_{-\infty}^{\infty} X(\xi) Y^*(\xi + t) d\xi)</td>
</tr>
<tr>
<td></td>
<td>(X \star Y = \hat{X}(\hat{Y})^*)</td>
</tr>
<tr>
<td>Autocorrelation-Energy spectrum</td>
<td>(\hat{X} \star \hat{X} =</td>
</tr>
<tr>
<td>Heisemberg uncertainty</td>
<td>(\sigma_t^2 \sigma_f^2 \geq \frac{1}{(2\pi)^2})</td>
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<tr>
<td></td>
<td>(\mu_T = \int_{-\infty}^{\infty} t F_T(t) dt), (\sigma_T^2 = \int_{-\infty}^{\infty} (t - \mu_T)^2 F_T(t) dt).</td>
</tr>
</tbody>
</table>

Spectral analysis of stochastic processes

Fixed \(\zeta = \zeta_0\): Function \(X(t) = X(t, \zeta_0)\).

Fixed \(t = t_0\): Random Variable \(X(\zeta) = X(t_0, \zeta)\).
## Two approaches to derive the spectrum of a process

**Goal:** estimate the spectrum of a stochastic process $X(t,\zeta)$.

**Means:** Fourier theory developed for "deterministic" functions.

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<tr>
<th>Approach 1 (simple)</th>
<th>Approach 2 (rigorous)</th>
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<tbody>
<tr>
<td>Process = family of realizations</td>
<td>Represent process $X(t)$ as $X(t) = \sum_{\alpha} H_\alpha = \sum_{\alpha} A_n e^{2\pi i f_\alpha t}$</td>
</tr>
<tr>
<td>Realization = function $X(t)$;</td>
<td>$H_\alpha$ = harmonic process</td>
</tr>
<tr>
<td>Apply Fourier theory to realization;</td>
<td>Spectrum of $X(t)$: $\sigma^2_\alpha = \sum_\alpha \sigma^2_\alpha$</td>
</tr>
<tr>
<td>Average over all realizations.</td>
<td>Physical meaning of spectrum?</td>
</tr>
</tbody>
</table>

$X(t)$ should be 2-order stationary (harmonic processes):

- Constant mean and variance
- Auto-correlation $R(t_1, t_2) = E\{X(t_1)X^*(t_2)\} = R(t_2 - t_1)$

### Approach 1, simple (1)

1. Treat one realization as a "deterministic" function $X(t)$;
2. Calculate its spectrum using Fourier theory above.

However, if process $X(t)$ is 2-order stationary, one realization

1. does not have a Fourier series representation ($X$ is not periodic)
2. does not have a Fourier integral representation since

$$\int_{-\infty}^{\infty} |X(t)|dt = \infty.$$

Try to circumvent this difficulty by using a limit process.
Approach 1, simple (2)

Limit process:

a) Window the function $X(t)$;
b) Define the spectrum of the windowed function
c) Let the window width go to $\infty$.

a) Windowed version of $X(t)$:

$$X_T(t) = \begin{cases} X(t), & -T/2 \leq t \leq T/2, \\ 0, & \text{otherwise} \end{cases}$$

$X_T$ has a Fourier transform

$$\hat{X}_T(f) = \int_{-\infty}^{\infty} X_T(t) \exp(-2\pi ift) \, dt = \int_{-T/2}^{T/2} X(t) \exp(-2\pi ift) \, dt$$
Approach 1, simple (3)

b) Apply Parseval relation for $X_T \leftrightarrow$ energy spectrum:

| Total energy: $\int_{-T/2}^{T/2} |X_T(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{X}_T(f)|^2 \, df$
| Energy in $[f, f+\Delta f]$: $|\hat{X}_T(f)|^2 \, df$

But energy of $X$ is infinite:
if $T \to \infty$, both total energy and spectrum $\to \infty$.

Approach 1, simple (4)

However,
- $X$ is 2-order stationary (variance is constant in time) $\Rightarrow$
- energy (or spectrum) $\sim$ window length $T$.
- Normalizing by $T$ should keep the values bounded;
- Energy/time=power, so

| Total power: $\frac{1}{T} \int_{-T/2}^{T/2} |X_T(t)|^2 \, dt = \frac{1}{T} \int_{-\infty}^{\infty} |\hat{X}_T(f)|^2 \, df$
| Power in $[f, f+\Delta f]$: $\frac{1}{T} |\hat{X}_T(f)|^2 \, df$

If $\lim_{T \to \infty} \frac{1}{T} |\hat{X}_T(f)|^2 < \infty$ then for $X(t)$

| Power in $[f, f+\Delta f]$: $\lim_{T \to \infty} \frac{1}{T} |\hat{X}_T(f)|^2 \, df$

Note: this is a time average!
c) Derive power spectrum for the process, by averaging over ensemble:

**Definition**

The power spectrum (power spectral density) of process $X(t)$ is

$$h(f) = \text{average of power in } [f, f + df] = \lim_{T \to \infty} \frac{1}{T} E \left\{ \left| \hat{X}_T(f) \right|^2 \right\}.$$  

**Theorem**

If $X(t)$ 2-order stationary process with auto-covariance $R(\tau)$, then $h(f)$ exists for all $f$, and

$$h(f) = \int_{-\infty}^{\infty} R(\tau) e^{-2\pi i ft} d\tau, \quad R(\tau) = \int_{-\infty}^{\infty} h(f) e^{2\pi i ft} df.$$  

**Auto-correlation** (deterministic function)

$$(X \ast X)(\tau) = \int_{-\infty}^{\infty} X(t)X^*(t + \tau) dt \quad \xrightarrow{\text{FT}} \quad \left| \hat{X}(f) \right|^2$$

Replaced by the **auto-correlation** for a stochastic process:

$$R(\tau) = E \{ X(t)X^*(t + \tau) \} \quad \xrightarrow{\text{FT}} \quad h(f) = \lim_{T \to \infty} \frac{1}{T} \left| \hat{X}_T(f) \right|^2$$

Equivalent to replacing ensemble by time average:

$$E \{ F(t) \} \longrightarrow \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} F(t) dt$$
Approach 1 summary

Steps to build the power spectrum of a stochastic process:

1. Take one representation $X(t)$ (no FS/FT representation);
2. Build truncated version $X_T(t)$ (has FT representation);
3. Define for $X_T(t)$ power spectrum $\left| \frac{1}{T} \hat{X}_T(f) \right|^2$, and let $T \to \infty$;
4. Average over the ensemble $h(f) = \text{E}\left\{ \lim_{T \to \infty} \left| \frac{1}{T} \hat{X}_T(f) \right|^2 \right\}$.

**Meaning:** $X_T(t) \mapsto \text{sum of harmonic oscillations } f \text{ of power } \left| \frac{1}{T} \hat{X}_T(f) \right|^2$

**Stochastic process:** $X_T(t) \mapsto \text{sum of harmonic processes of power } h(f)$.
We built this as a limiting process.
**But is there such a decomposition for a given process $X(t)$?**

Approach 2, rigorous

**Theorem**

Let $X(t)$ be a 2-order stationary process. There exists an orthogonal process $Z(f)$ such that

$$X(t) = \int_{-\infty}^{\infty} e^{2\pi i ft} dZ(f),$$

with the properties

- $\text{E}[dZ(f)] = 0$
- $\text{E}\left[|dZ(f)|^2\right] = dH(f)$ with $dH(f) = \text{power contributed by } [f, f + df]$.
- For any $f_1 \neq f_2$, $\text{E}[dZ(f_1)dZ^*(f_2)] = 0$.

(e.g. Priestley, MB, Spectral Analysis and Time Series)

- Integral is Stieltjes; convergence in the MS sense.
- continuous spectrum: $dH(f) = H'(f) df = h(f) df$, $h(f) = \text{power spectrum}$.
- Any 2-order stationary process = sum of harmonic processes with uncorrelated amplitudes and phases.
Example 1. White noise

For the white-noise process:

\[ C(\tau) = q \delta(\tau), \text{ with } q \geq 0. \]

Power spectrum

\[ h(f) = F[C] = q \int_{-\infty}^{\infty} \delta(\tau) \exp(-2\pi if \tau) \, d\tau = q \geq 0. \]

All frequencies have equal power.

Example 2. Harmonic process

Definition

\[ H(t) = \sum_{n=1}^{N} A_n \cos(\omega_n t + T_n); A_n \text{ and } T_n \text{ mutually uncorrelated, } T_n \text{ uniformly distributed in the interval } [-\pi, \pi]. \]

- auto-covariance

\[ C(\tau) = \sum_{n} \frac{1}{2} \mathbb{E} \{ A_n^2 \} \cos(2\pi f_n \tau), \]

- power spectral density

\[ h(f) = \frac{1}{4} \sum_{n} \mathbb{E} \{ A_n^2 \} \left[ \delta(f + f_n) + \delta(f - f_n) \right]. \]

- “one-sided” spectrum

\[ h(f) = \sum_{n} \frac{1}{2} \mathbb{E} \{ A_n^2 \} \delta(f - f_n), \text{ with } f_n \geq 0. \]
Fourier Analysis
Linear transformations and filters
DETERMINISTIC PERIODIC Functions: Fourier Series
DETERMINISTIC APERIODIC Functions: Fourier Integrals
2-Order Stationary Random Process

One Realization vs. Process

\[
\frac{1}{T} |\hat{X}(f)|^2 \neq h(f)
\]

Note: Covariance-Ergodic processes

Previously, we replaced the ensemble average by the time average.

\[
M = E \{G(x)\} \longrightarrow M = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} G(x) \, dt.
\]

Definition

Processes which have the property that

\[
E \{G(x)\} = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} G(x) \, dt
\]

are called “M–ergodic”.

Theorem

A zero-mean stochastic process is covariance-ergodic if \(\lim_{\tau \to \infty} C(\tau) = 0\)
Systems and process transformation

A procedure that transforms process $Y(t)$ into process $X(t)$

- In the process, some characteristics of input $Y$ are changed. Output $X$ is different.
- Spectra of $Y$ and $X$ are different. Some frequencies have been “filtered out”.

Linear system/transformation

$X(t)$ and $Y(t)$ are two stochastic processes.

**Definition**

A system (or transformation) $L$ maps input $Y(t)$ into output $X(t)$:

$$X(t) = L[Y(t)]$$

$L$ is an operator acting only on functions of $t$.

$L$ is **linear** if $a, b \in \mathbb{C}$

$$L[aY_1(t) + bY_2(t)] = aL[Y_1(t)] + bL[Y_2(t)].$$

$L$ is **shift-invariant** if for any $s \in \mathbb{R}$

$$X(t) = L[Y(t)] \Rightarrow X(t - s) = L[Y(t - s)].$$

Transformations discussed here are assumed linear and shift-invariant.
Examples: General systems

Let $Y(t)$ and $Y(t)$ be discrete-time processes: $Y(t) = Y(n\Delta t)$ and $X(t) = X(n\Delta t)$.

Two (3) basic ways to combine them:

1. Autoregressive model AR(n):
   
   $$a_0 X(t) + a_1 X(t - \Delta t) + a_2 X(t - 2\Delta t) + \ldots + a_n X(t - n\Delta t) = Y(t)$$

2. Moving average model MA(n):
   
   $$X(t) = b_0 Y(t) + b_1 Y_1(t - \Delta t) + b_2 Y(t - 2\Delta t) + \ldots b_n Y(t - n\Delta t)$$

3. Autoregressive Moving Average ARMA(m,n)
   
   $$\sum_{k=0}^{n} a_k X(t - k\Delta t) = \sum_{k=0}^{m} b_k Y(t - k\Delta t)$$

Examples: Autoregressive model AR

Second order discrete AR(2):

$$a_0 X(t) + a_1 X(t - \Delta t) + a_2 X(t - 2\Delta t) = Y(t),$$

- Output given through an implicit relationship.
- Physically realizable condition: only $X(t - n\Delta t)$.

Second order continuous AR(2): Use

$$\Delta X(t) = X(t) - X(t - \Delta t),$$

$$\Delta^2 X(t) = \Delta X(t) - \Delta X(t - \Delta t) = X(t) - 2X(t - \Delta t) + X(t - 2\Delta t)$$

Let $\Delta t \to 0$

$$\frac{d^2 X(t)}{dt^2} + \alpha_1 \frac{dX(t)}{dt} + \alpha_2 X(t) = Z(t).$$

(Bumpy road - forced oscillation model)
Examples: AR(n)

Autoregressive process of arbitrary order AR(n):

\[ a_0 X(t) + a_1 X(t - \Delta t) + a_2 X(t - 2\Delta t) + \ldots + a_n X(t - n\Delta t) = X(t), \]

and

\[ a_0 \frac{d^n X(t)}{dt^n} + a_1 \frac{d^{n-1} X(t)}{dt^{n-1}} + \ldots + a_n \frac{dX(t)}{dt} + a_n X(t) = Z(t), \]

Examples: Moving Average model (MA)

Discrete Moving Average MA(n):

\[ X(t) = b_0 Y(t) + b_1 Y(t - \Delta t) + b_2 Y(t - 2\Delta t) + \ldots b_n Y(t - n\Delta t), \]

Continuous Moving Average MA:

Replace \( b_j \) by function \( w(t) \) (window)

\[
\begin{align*}
  w(t) & \neq 0 & 0 \leq s \leq T \\
  w(t) & = 0 & \text{otherwise}
\end{align*}
\]

MA is the convolution of \( Y(t) \) with \( w \):

\[ X(t) = \int_{-\infty}^{\infty} Y(s) w(t-s) \, ds = Y \circ w. \]

Note: convolution with window \( w \) has also the effect of “smearing” the function \( Y \).
Examples: General ARMA (continuous)

For continuous-time processes, the general form of a linear system:

\[
\alpha_0 \frac{d^n X(t)}{dt^n} + \alpha_1 \frac{d^{n-1} X(t)}{dt^{n-1}} + \ldots + \alpha_{n-1} \frac{dX(t)}{dt} + \alpha_n X(t) = \int_{-\infty}^{\infty} w(t-s)Y(s)ds,
\]

where \( \alpha_j \in \mathbb{C} \) (\( j = 1, 2, \ldots, n \)) and \( w(t) \) is a window function,

\[
\int_{-\infty}^{\infty} w(t)dt = 1.
\]

Fundamental Theorem

The fundamental theorem states:

If \( L \) is a shift-invariant linear transformation

\[
X(t) = Y(t) \circ G(t) = \int_{-\infty}^{\infty} Y(s)G(t - s)ds,
\]

\( G(t) = L[\delta(t)] = \text{impulse response of } L \)

A linear system is completely characterized by its impulse response. Steps:

1. Obtain the function \( G(t) = \text{response of } L \) to a delta impulse,
2. Calculate output \( X \) as convolution of input \( Y \) and \( G \).
Linear transformations of moments

Mean
$L$ and $E \{ . \}$ are linear operators and commute (acting on independent variables), therefore

$$\mu_X(t) = E[X(t)] = E[L[Y(t)]] = L[E[Y(t)]] = L[\mu_Y(t)].$$

Auto correlation
Let $L_{t_2} = \text{linear system operates on the variable } t_2$. Regard $t_1 = \text{parameter.}$ Then

$$R_{YX}(t_1, t_2) = L_{t_2}[R_{YY}(t_1, t_2)] = R_{YY}(t_1, t_2) \circ G(t_2);$$
$$R_{XX}(t_1, t_2) = L_{t_1}[R_{YX}(t_1, t_2)] = R_{YX}(t_1, t_2) \circ G(t_2);$$

where, for example,

$$R_{YY}(t_1, t_2) \circ G(t_2) = \int_{-\infty}^{\infty} R_{YY}(t_1, t_2 - s) G(s) \, ds.$$ 

Similar relations for auto-/cross-covariances.

Example: White noise

Input $Y(t)$ is a zero-mean, white-noise process

$$C_{YY}(t_1, t_2) = q(t_1) \delta(t_2 - t_1), \quad q(t_1) \geq 0.$$

Power of output $X(t)$

$$E \left[ |X(t)|^2 \right] = q(t) \circ |G(t)|^2 = \int_{-\infty}^{\infty} q(t - s) |G(s)|^2 \, ds.$$

If $Y(t)$ is stationary $q(t) = q$,

$$E \left[ |X(t)|^2 \right] = qE, \quad \text{where } E = \int_{-\infty}^{\infty} |G(s)|^2 \, ds.$$
Goal: power spectrum transformation

<table>
<thead>
<tr>
<th>Time domain</th>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{YY}$</td>
<td>$R_{YY} \circ G$</td>
<td>$R_{YX}$</td>
</tr>
<tr>
<td>$R_{YY}$</td>
<td>$R_{YX} \circ G$</td>
<td>$R_{XX}$</td>
</tr>
<tr>
<td>FT</td>
<td>FT</td>
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<table>
<thead>
<tr>
<th>Frequency domain</th>
<th>$h_{YY}$</th>
<th>$h_{YX}$</th>
<th>$h_{XX}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_{YY}$</td>
<td>$h_{YX}$</td>
<td>$h_{XX}$</td>
<td></td>
</tr>
</tbody>
</table>

Power spectrum transformation (1)

Assume that $X(t)$ and $Y(t)$ are 2-order stationary:

- Constant mean and variance,
- Auto-correlation $R(t_1, t_2) = R(t_2 - t_1) = R(\tau)$.

**Theorem**

The following statements are true for a shift-invariant linear system:

$$
R_{YX}(\tau) = R_{YY}(\tau) \circ G^*(-\tau), \quad R_{XX}(\tau) = R_{YX}(\tau) \circ G(\tau),
$$

$$
h_{YX}(f) = h_{YY}(f) \hat{G}^*(f), \quad h_{XX}(f) = h_{YX}(f) \hat{G}(f),
$$

Where

- $\hat{G}(f) = \mathcal{F}[G(t)];$
- $h_{YX}(f) = \mathcal{F}[R_{YX}(t)] = \text{cross-spectrum of } X \text{ and } Y;$
- $h_{XX} = \mathcal{F}[R_{XX}(t)] = \text{(auto) spectrum of } X.$
**Theorem**

**Combining equations:**

\[
R_{XX}(\tau) = R_{YY}(\tau) \circ G(\tau) \circ G^*(-\tau) = R_{YY}(\tau) \circ [G(\tau) * G(\tau)],
\]

\[
h_{XX}(f) = h_{YY}(f) |\hat{G}(f)|^2.
\]

**Conclusions:**

- Spectrum of output = spectrum of input $\times$ factor function $|\hat{G}(f)|^2$.
- "Factor function" defined by impulse response function of $L$. 
Input is zero-mean white noise with $\delta$ auto-correlation and constant spectral distribution:

$$R_{YY}(\tau) = q\delta(\tau), \quad h_{YY}(f) = q,$$

Output spectrum $\sim$ square of window spectrum

$$h_{XX}(f) = q \left| \hat{G}(f) \right|^2, \quad R_{XX}(\tau) = q \left[ G(\tau) \ast G(\tau) \right].$$
Examples: Continuous AR(1)

A continuous AR(1) process is described by equation

\[ \frac{dX(t)}{dt} + aX(t) = Y(t), \]

Impulse response function \( G(t) \)

\[ \frac{dG(t)}{dt} + aG(t) = \delta(t), \]

\[ h_{XX}(f) = h_{YY}(f) \left| \hat{G}(f) \right|^2 = \frac{h_{XX}(f)}{a^2 + \omega^2}, \quad \omega = 2\pi f. \]

If input is a white-noise, \( h_{YY}(f) = q \), the output spectrum is

\[ h_{XX}(f) = \frac{q}{a^2 + \omega^2} \geq 0. \]

Examples: Continuous MA

System is a moving average with window \( w \)

\[ X(t) = \int_{-\infty}^{\infty} Y(s) w(t - s) \, ds, \quad \text{with} \quad \int_{-\infty}^{\infty} w(t) \, dt = 1. \]

Impulse response:

\[ G(t) = L[\delta(t)] = \int_{-\infty}^{\infty} \delta(s) w(t - s) \, ds = w(t). \]

\[ h_{XX}(f) = h_{YY}(f) |\hat{w}(f)|^2 \]

If \( w \) is a rectangular window, its FT is a cardinal sine:

\[ w(t) = \begin{cases} 
1/T & |t| \leq T/2 \\
0 & \text{otherwise}
\end{cases}; \quad \hat{w}(f) = \frac{2a \sin(\pi f T)}{\pi f T} = \text{sinc}(\pi f T). \]

If input is white noise:

\[ h_{XX}(f) = q |\text{sinc}(\pi f T)|^2. \]
AR(1) & MA spectral window

AR(1): \( h_{XX}(f) = h_{YY}(f) \left| \hat{G}(f) \right|^2; \quad \left| \hat{G}(f) \right|^2 = \frac{1}{\sigma^2 + \omega^2}, \)

MA: \( h_{XX}(f) = h_{YY}(f) \left| \hat{G}(f) \right|^2; \quad \left| \hat{G}(f) \right|^2 = h_{YY}(f) \left| \text{sinc}(\pi f T) \right|^2 \)

MA and AR behave like low-pass filters.

Filters

Relations used as filters

\[
X(t) = \int_{-\infty}^{\infty} Y(s) G(t - s) \, ds,
\]
\[
h_{XX}(f) = h_{YY}(f) \left| \hat{G}(f) \right|^2;
\]

Motivation: Design \( G(t) \) so that \( \left| \hat{G}(f) \right|^2 \approx 0 \) for some frequencies.

Impulse response: \( G(t) \), completely characterizes the filter.

Transfer function: \( H(f) = \hat{G}(f) \) complex, \( H(f) = \rho(f) e^{i\phi(f)} \).

Gain: Modulus \( \rho(f) \) = amplification @ \( f \).

Phase: Angle \( \phi(f) \) = phase shift @ \( f \).